Σ^1_3-Absoluteness in Forcing Extensions

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Abstract

We investigate the consistency strength of the forcing axiom for Σ^1_3 formulas, for various classes of forcings. We review that the consistency strength of Σ^1_3-absoluteness for all set forcing or even just for ω_1-preserving forcing is that of a reflecting cardinal. To get the same strength from the forcing axiom restricted to proper forcing, one can add the hypotheses that ω_1 is inaccessible to reals. Then we investigate the strength of the forcing axiom restricted to *ccc* forcing notions under this additional hypothesis; to gauge it we introduce a weak version of a weak compact cardinal, namely, a lightface Σ^1_2-indescribable cardinal.

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1 Preliminary facts

This section contains some definitions, well known facts and basic theorems which we will use throughout the present paper.

First, we take a little time to fix our notation (we hope to include all possible sources of confusion). For a structure $\mathcal{M} = \langle M, \in; P_0, \ldots, P_k\rangle$, $\mathcal{L}^{\mathcal{M}}$ denotes the language $\mathcal{L} = \{\in, P_0, \ldots, P_k\}$. $\mathcal{L}^{\mathcal{M}}_X$ denotes the same language with constants for all the elements of X. We write $\mathcal{M} \prec \mathcal{N}$ to denote elementarity (i.e. $\mathcal{M}$ and $\mathcal{N}$ have the same theory in $\mathcal{L}^{\mathcal{M}}_M$), and we write $\pi : \mathcal{M} \to_{\Sigma_\omega} \mathcal{N}$ to express π is an elementary embedding (i.e. $\langle ran(\pi), \pi'' \in ; \pi'' P_0, \ldots, \pi'' P_k\rangle \prec \mathcal{N}$). For transitive models, we call the least ordinal $\alpha \in \mathcal{M}$ such that $\pi(\alpha) > \alpha$ the *critical point of* π, and denote it by $crit(\pi)$. For a structure $\mathcal{N} = \langle N, E, \ldots\rangle$ such that E is extensional and well-founded (and, in the case of a class sized structure, set-like), we denote its transitive collapse (that is, the unique structure $\langle M, \in, \ldots\rangle$ isomorphic to $\mathcal{N}$ and such that M is transitive) by $tcoll(\mathcal{N})$. By $(\phi)^{\mathcal{M}}$ we mean the formula ϕ holds relativized to a (possibly class-sized) structure $\mathcal{M}$ and by $t^{\mathcal{M}}$ we mean the interpretation of a term t as from the viewpoint of $\mathcal{M}$. When we talk about structures $\mathcal{M} = \langle M, \in; P_0, \ldots, P_k\rangle$, we sometimes omit listing the $\in$ as a predicate. When we use notation that is defined for structures in a context when we talk about sets M, it is implied we mean the structure $\langle M, \in\rangle$.

We denote the usual Zermelo-Fraenkel axioms by ZF, and that theory with AC (the axiom of choice) added by ZFC. ZF^- and ZFC^- denote the corresponding theories with the power set axiom deleted.

When we talk about Skolem functions F_ϕ for a (possibly class-sized) structure $\mathcal{M}$, we consider $\mathcal{M}$ to be equipped with a well-ordering $<_{\mathcal{M}}$. Then, if $\phi(x_0, x_1)$ is a formula of $\mathcal{L}^{\mathcal{M}}$ with two free variables, for $p \in M$, we define $F_\phi(p)$ to be the $<_{\mathcal{M}}$-least $y \in M$ such that $\mathcal{M} \vDash \phi(y, p)$ if such y exists, and to be $\emptyset$ otherwise. For $X \subseteq M$, we denote by $h^{\mathcal{M}}_{\Sigma_\omega}(X)$ the *Skolem hull of X in $\mathcal{N}$*, that is the smallest (in the sense of $\subseteq$) set $Y \subseteq M$ s.t. $Y \supseteq X$ that is closed under all Skolem functions F_ϕ, for ϕ a formula of $\mathcal{L}^{\mathcal{M}}$.

We use the familiar $\Sigma_k, \Pi_k, \Delta_k$ notation for the classes formulas that can be written with k blocks of equal quantifiers starting with $\exists$ or $\forall$ and for the intersection of these first two classes, respectively. We shall give a more thorough review of the analytical hierarchy in section 1.3; in section 5.1 we quickly review the general Σ^k_n hierarchies of higher order formulas over a structure.

We write Lim, $Card$, Reg for the class of limit ordinals, cardinals and regular cardinals respectively. For a set x, $|x|$ is its cardinality, and we sometimes write $x \lesssim y$ to mean that $|x| < |y|$. $TC(x)$ denotes the transitive closure of x. For any cardinal κ, let H_κ denote the set of all x such that

$TC(x)$ has size less than κ (i.e., the H_κ hierarchy is continuously defined also at singular cardinals). HC denotes H_{ω_1}, the set of hereditarily countable sets.

As far as forcing is concerned, we sometimes write V^P for a generic extension $V[G]$, where G is P-generic over V, if the particular G concerned is of no importance. We write dots over names (e.g. $\dot{q}$) and $\dot{q}[G]$ for the interpretation of the name $\dot{q}$ by a generic G. For sets a in the ground model we write $\check{a}$ for the "standard name" for a.

1.1 General set theory

1.1 Definition. 1. $C \subset [A]^\omega$ is called *cub* if, and only if, there are predicates $P_1, \ldots, P_k \subset A$ such that

$$C = \{X \in [A]^\omega \mid \langle X, P_1, \ldots, P_k \rangle \prec \langle A, P_1, \ldots, P_k \rangle\}$$

2. $S \subset [A]^\omega$ is called *stationary* if, and only if,

$$\text{for all } cub \text{ sets } C \subset [A]^\omega,\ S \cap C \neq \emptyset$$

1.2 Fact. Let $\mathcal{N} = \langle N, R_0, \ldots R_k \rangle$, $X \in H_\kappa$ and $X \subseteq N$. Then there exists a transitive $M \in H_\kappa$ s.t. $X \subseteq M$ together with $R'_0, \ldots R'_k \subseteq M$, and an elementary embedding $i : \langle M, R'_0, \ldots R'_k \rangle \to_{\Sigma_\omega} \langle N, R_0, \ldots R_k \rangle$ s.t. i is the identity on X.

Proof. First observe that since $X \in H_\kappa$, also $TC(X) \in H_\kappa$, whence we can assume X to be transitive. Now start by taking the Skolem-hull (with respect to the language where the additional predicates have been added): let $\bar{M} := h^{\mathcal{N}}_{\Sigma_\omega}(X)$. $\bar{M}$ has size less than κ. Set $M := tcoll(\bar{M})$, the transitive collapse of $\bar{M}$. Being transitive and of size less than κ, $M \in H_\kappa$. Let i be the inverse of the collapsing map (the new predicates are just the pullbacks of the original predicates). As $X \subseteq \bar{M}$ was transitive, i is the identity on X. $\square$

1.3 Fact. If N is transitive and $\lambda + 1 \subseteq M \prec N$, then $(H_{\lambda^+})^M$ is transitive.

Proof. Let $x \in (H_{\lambda^+})^M$ be arbitrary, we show $x \subseteq M$. $M \models$"$\exists f : \lambda \twoheadrightarrow TC(x)$", and so by elementarity, for some $f \in M$, $N \models$"$f : \lambda \twoheadrightarrow TC(x)$". Now for arbitrary $y \in x$, we have $y = f(\xi)$ for some $\xi \in \lambda$, while $\lambda \subseteq M$. It is easily checked that $f, \xi \in M$ by elementarity implies $f(\xi) \in M$, so $y \in M$. $\square$

1.4 Corollary. *Let A be any set of ordinals, κ a successor cardinal. Then $C := \{\ \alpha < \kappa \mid L_\alpha[A] \prec L_\kappa[A]\ \}$ is cub in κ.*

Proof. For $\xi < \kappa$, let $\chi(\xi)$ denote the least ordinal such that Skolem functions for $L_\kappa[A]$ on $L_\xi[A]$ have values in $L_{\chi(\xi)}[A]$. Then $\chi : \kappa \to \kappa$ and C is precisely the set of closure points of χ, hence *cub*. $\square$

1.5 Fact. If $M \vDash ZF^-$ is transitive, $\pi : M \to_{\Sigma_\omega} N$ an elementary embedding with critical point α, then $\alpha \in Reg^M$ and $\pi \restriction {H_\alpha}^M = id$.

Proof. Assume M thinks α is singular; then there is a function f in M, with domain an ordinal below α and values below α whose range is cofinal in α. But as all these ordinals are not moved by π, $\pi(f) = f$ and thus $\alpha = sup(ran(f)) = sup(ran(\pi(f))) = \pi(sup(ran(f))) = \pi(\alpha)$, contradiction. As $\pi \restriction \alpha = id$, $\alpha \subseteq ran(\pi)$. By hypothesis $ran(\pi) \prec N$, we can apply 1.3: for each $\lambda < \alpha$, $(H_{\lambda^+})^{ran(\pi)}$ is transitive. So $(H_{\pi(\alpha)})^{ran(\pi)} = \pi''(H_\alpha)^M$ (as the union of these transitive sets) is seen to be transitive. Note that as the transitive collapse of any set A is uniquely defined as the transitive set isomorphic to A, we have that M must be the transitive collapse of $ran(\pi)$ and π must be the inverse of the collapsing map. As the collapsing map is just the identity on any transitive set, π is the identity on $(H_\alpha)^M$. $\square$

1.6 Fact. Let $\mathcal{M} = \langle M, R_1, \ldots, R_k \rangle$, $\mathcal{N} = \langle N, S_1, \ldots, S_r \rangle$, be models such that $k \le r$, R_i has the same arity as S_i for $i \le k$, and $\mathcal{M}$ is countable. Then the statement

$$\exists \pi, \exists R_{k+1}, \ldots, R_r \quad \pi : \langle M, R_1, \ldots, R_r \rangle \to_{\Sigma_\omega} \langle N, S_1, \ldots, S_r \rangle \tag{1}$$

is absolute for transitive models U of ZF^- such that U contains both $\mathcal{N}$ and $\mathcal{M}$ and $U \vDash \mathcal{M} \cong \omega$.

Proof. Let $m : \omega \to M$ be an enumeration of M, $(\phi_i)_{i \in \omega}$ an enumeration of formulas of the language of $\mathcal{N}$. Let $\gamma : \omega \to \omega$, $\delta : \omega \to {}^{<\omega}\omega$ be such that together, they enumerate all formulas of the language $\mathcal{L}_M^{\mathcal{N}}$ (the language of $\mathcal{N}$ with constants for all the elements of M) in the following sense: for all n, $ran(\delta(n)) \subseteq n$, the number of free variables of $\phi_{\gamma(n)}$ is no larger than $dom(\delta(n))$, and γ and δ are onto in the sense that $\phi_{\gamma(n)}(m(s(0)), \ldots, m(s(l)))$, for $s = \delta(n)$ and $l = lh(s) - 1$, runs through all formulas of $\mathcal{L}_M^{\mathcal{N}}$ as n runs through ω. Think of the elementary embedding as an interpretation of the constant symbols of $\mathcal{L}_M^{\mathcal{N}}$.

We inductively define a tree T searching for the embedding and the new predicates. Let M_k denote the set $\{m(0), \ldots, m(k-1)\}$. We let $f \in T_{n+1}$, the $n+1$-th level of T, if and only if $f : n+1 \to N$ such that

- $f \restriction n \in T_n$
- $f \circ m^{-1} : \langle M_{n+1}; R_1, \ldots, R_k \rangle \to \langle N; S_1, \ldots, S_k \rangle$ is a homomorphism

- setting $s = \delta(n)$ and $l = lh(s)-1$, if $N \vDash \exists x \phi_{\gamma(n)}(x, f(s(0)), \ldots, f(s(l)))$ then $f(n)$ is such that $\phi_{\gamma(n)}(f(n), f(s(0)), \ldots, f(s(l)))$. Otherwise, $f(n)$ is allowed to be an arbitrary element of N.

If $(f_i)_{i\in\omega}$ is an infinite branch through T, $\pi := (\bigcup_{i\in\omega} f_i)\circ m^{-1}$ is an elementary embedding of $\langle M, R_1, \ldots, R_r\rangle$ into $\mathcal{N}$, where the additional predicates are obtained by taking the (componentwise) pre-image of the predicates of $\mathcal{N}$ under π. On the other hand, any such relations together with an embedding define an infinite branch through T. So (1) is equivalent to the statement

$$\supseteq \text{ is not well-founded on } T \tag{2}$$

Any U as in the hypothesis contains T, whose definition is Δ_1 and thus absolute. Then by juggling infinite branches and ranking functions for T, (2) is seen to be absolute. □

1.2 Forcing

This section reviews the basic properties of some well-known forcing notions that are used in this paper.

General forcing facts

We use the largely standardized forcing notation as developed for partial orders; we shall take for granted basic facts about forcing and finite iterations as found in texts such as [Kun80]. We shall only venture under the hood of the forcing machinery on one occasion, namely to describe what is known as *universality of the Lévy-Collapse* (corollary 1.30). To this end, we repeat some basic facts. A proof of the next fact can be found in [Kun80, p. 220f.]

1.7 Definition. Let P, Q be p.o.'s. A map $i : P \to Q$ is called a *complete embedding* if i is order-preserving and

1. $\forall p, q \in P \quad i(p) \| i(q) \;\Rightarrow\; p \| q$
2. $\forall q \in Q \quad \exists p_0 \in P$ s.t. $\forall p \in P \quad p \leq p_0 \;\Rightarrow\; i(p) \| q$

A map $i : P \to Q$ is called a *dense embedding* if it is order preserving and $i''P$ is dense in Q. Let $\sim$ be the least equivalence relation of partially ordered sets extending the relation "there exists a dense embedding from P into Q". Equivalently, $P \sim Q$ if $r.o.(P)$ and $r.o.(Q)$ are the same (up to isomorphism) i.e. they are canonically associated with the same Boolean algebra (see [II,3.3, p.62][Kun80] or [Jec78, lemma 17.2, p.152]). If $P \sim Q$, they are called *equivalent*.

1.8 Fact. 1. Let $i : P \to Q$ be a complete embedding. Then if H is Q-generic, $G := (i^{-1})''H$ is P-generic and $V[G] \subseteq V[H]$.

2. Let $d : P \to Q$ be a dense embedding. Then not only is d complete and the above holds, but for any P-generic G, $H := \{q \in Q \mid \exists p \in P \quad d(p) \leq q\}$ is Q-generic and $V[G] = V[H]$. If $P \sim Q$, P and Q yield the same extensions and a generic for either can be used to construct a generic for the other.

1.9 Fact. Suppose P is a p.o., $\dot{Q}$ a P-name, $\Vdash_P$"$\dot{Q}$ is a p.o." and M a transitive model of ZF (and let p_i denote the canonical projection to the i-th coordinate). Then:

1. If I is generic for $P * \dot{Q}$ over M, $G := \{p | \exists \dot{q}(p, \dot{q}) \in I\}$ is P-generic over M, $H := I[G]$ is $\dot{Q}[G]$-generic over $M[G]$ and $M[I] = M[G][H]$.

2. If G is P-generic over M and H is $\dot{Q}[G]$-generic over $M[G]$,

$$G * H := \{(p, \dot{q}) \in P * \dot{Q} | \ p \in G \ \wedge \ \dot{q}[G] \in H \ \}$$

   is $P * \dot{Q}$-generic over M and $M[G * H] = M[G][H]$.

Moreover, $i_P : P \to P * \dot{Q}$, defined by $p \mapsto (p, \dot{1}_Q)$ is a complete embedding of P into $P * \dot{Q}$. For $Q \in M$, all of the above holds for $P \times Q$, $G \times H$ and $H := \{q | \exists p(p, q) \in I\}$, and i_Q (defined in complete analogy to i_P) is a complete embedding.

Almost disjoint coding

Let α a regular cardinal, $\beta > \alpha$ some ordinal. Let $\mathcal{A} = (a_\xi)_{\xi<\beta}$ be a family of unbounded subsets of α such that for $\xi \neq \xi' < \beta$, $|a_\xi \cap a_{\xi'}| < \alpha$. This is called an almost disjoint (or a.d.) family on α. Further, let B be any subset of β. Using $\mathcal{A}$ we can force to add a subset A of α, such that A codes B in the following sense:

$$B = \{\xi < \beta | \quad |A \cap a_\xi| < \alpha\} \tag{3}$$

We shall call this forcing $P_{\mathcal{A},B}$, the almost disjoint coding of B using $\mathcal{A}$. Of course this method of adding a set coding B depends on the existence of an appropriate a.d. family. For basic results about constructing a.d. families, see [Kun80, p. 47]. We will use the easy facts that there always exists an a.d. family on α of size α^+ (by a maximality argument using Zorn's lemma), and that there is an a.d. family on ω of size 2^ω (just take sets of code numbers of finite strings approximating a real - the same works with ω replaced by some strong limit cardinal). We shall use the notation of the preceding paragraph throughout this section without repeating the conditions imposed on, e.g., α and β. For the moment, let $\mathcal{A} \restriction q$ denote $\{a_\xi \mid \xi \in q\}$.

1.10 Definition. Almost Disjoint Coding. Define

$$P_{\mathcal{A},B} := [\alpha]^{<\alpha} \times [B]^{<\alpha}$$

ordered by

$$(p,q) \leq (\bar{p},\bar{q}) \iff p \text{ end-extends } \bar{p},\ q \supseteq \bar{q}, \text{ and } (p \setminus \bar{p}) \cap \bigcup \mathcal{A} \restriction \bar{q} = \emptyset$$

The forcing consists of pairs, the first part of which is a an approximation of the set A to be added, the second part indexes the a_ξ we wish to avoid by further approximations. We must now show that the generic has the property promised in (3).

1.11 Lemma. *For each $\sigma \in B$, the set $D_\sigma := \{(p,q) \in P_{\mathcal{A},B} \mid \sigma \in q\}$ is dense in $P_{\mathcal{A},B}$.*

Proof. Given (p_0, q_0), just extend it to $(p_0, q_0 \cup \{\sigma\})$ to hit D_σ. $\square$

1.12 Lemma. *For each $\rho < \alpha$, $\sigma \in (\beta - B)$, the set*

$$D_{\rho,\sigma} := \{(p,q) \in P_{\mathcal{A},B} \mid p \cap a_\sigma \textit{ has order type at least } \rho\}$$

is dense in $P_{\mathcal{A},B}$.

Proof. Let (p_0, q_0) be a condition, ρ and σ given as above. Look at $S := a_\sigma - \bigcup \mathcal{A} \restriction q_0 = a_\sigma - [\bigcup_{\xi \in q_0}(a_\sigma \cap a_\xi)]$. As $\sigma \notin B$, for $\xi \in q_0$, $a_\sigma \cap a_\xi \lnsim \alpha$. Thus S is obtained by taking away the union of less then α sets of size less then α from a set of size α, and thus has size α, by regularity of α. Add a subset of S of order type ρ to p_0 and call the set so obtained p_1. As we have extended inside a_σ, $a_\sigma \cap p_1$ will have the desired order type, and as we have avoided all the a_ξ indexed by q_0, (p_1, q_0) is a condition stronger than (p_0, q_0). $\square$

Now let G be generic for $P_{\mathcal{A},B}$, and let $A := \bigcup\{p \mid (p,q) \in G, \text{some q}\}$. As G meets all of the above dense sets, A codes B in the sense of (3).

1.13 Fact. $P_{\mathcal{A},B}$ is α-closed.

Proof. Let $\rho < \alpha$, $(p_\xi, q_\xi)_{\xi < \rho}$ a decreasing sequence of conditions. Consider p' and q', the union of the first and second parts of the conditions in this sequence, respectively. (p', q') is a condition extending the conditions of the sequence: let $\xi \in q_\lambda$, $\lambda < \rho$. Then $p' \cap a_\xi = p_\lambda \cap a_\xi$. So $(p', q') \leq (p_\lambda, q_\lambda)$. $\square$

1.14 Fact. $P_{\mathcal{A},B}$ is $([\alpha]^{<\alpha})$-linked. If $[\alpha]^{<\alpha} = \alpha$, $P_{\mathcal{A},B}$ is α-centered.

Proof. Recall a forcing P is called λ-centered if there is $f : P \longrightarrow \lambda$ s.t. $\forall \xi < \lambda$, and $\forall W \in [f^{-1}(\xi)]^{<\lambda}$, there exists $\bigwedge W$, a weakest condition stronger than all the conditions in W. A forcing is λ-linked if there is such a function s.t. if $f(p) = f(p')$, p and p' are compatible. If $[\alpha]^{<\alpha} = \alpha$, let $f : P_{\mathcal{A},B} \longrightarrow [\alpha]^{<\alpha} \cong \alpha$ be the projection to the first part. For a set of conditions $W \lesssim \alpha$ with the same first part p_0, $(p_0, \bigcup_{(p,q)\in W} q)$ is $\bigwedge W$. If $[\alpha]^{<\alpha} > \alpha$, we can't find $\bigwedge W$ for $W \cong [\alpha]^{<\alpha}$, but $P_{\mathcal{A},B}$ will at least be linked, as conditions with the same first part are compatible. □

Note that λ-centeredness implies the λ-cc, as any two conditions with the same image under f must be compatible (centered implies linked). So if α is a strong limit (or ω), $P_{\mathcal{A},B}$ has the α^+-cc

Another kind of almost disjoint coding

We have seen that by forcing with $P_{\mathcal{A},B}$, we can code some given object of size 2^ω in the ground model by a single, generic real. Basically, we could use this forcing to code a function $f : 2^\omega \to 2^\omega$. For example, we could use a canonical bijection $g : 2^\omega \times 2^\omega \to 2^\omega$ and code the graph of f into a set of reals B before forcing with $P_{\mathcal{A},B}$.

What if we want to code a function $f : 2^\omega \to 2^\omega$ into a real A such that for any model M with sufficient closure properties and containing A and $\mathcal{A}$, for any real $r \in M$, we also have $f(r) \in M$? That is, we want to make sure that $f(r)$ can be decoded inside M from A. Using method described above, this will in general not be the case.

We will use a special almost disjoint family. Fix some arithmetical enumeration $(s_n)_{n\in\omega}$ of ${}^{<\omega}\omega$, and some arithmetical partition $(H_i)_{i\in\omega}$ of ω, where each H_i is infinite. For $r \in 2^\omega$, $i \in \omega$, let

$$a_r^i := \{n \in \omega \quad | \quad r \restriction lh(s_n) = s_n \quad \wedge \quad lh(s_n) \in H_i\}, \text{ and}$$
$$a_r := \bigcup_{i\in\omega} a_r^i = \{n \in \omega \quad | \quad r \restriction lh(s_n) = s_n\}$$

Clearly, $\{a_r^i \mid r \in 2^\omega,\ i \in \omega\}$ is an a.d.-family: let $(r, i) \neq (s, j) \in 2^\omega \times \omega$. If $i \neq j$, $a_r^i \cap a_s^j = \emptyset$. Otherwise $s \neq r$. Note that $a_r^i \subseteq a_r$, $a_s^j \subseteq a_s$. Assume $k_0 \in r\Delta s$; then for all $k \geq k_0$, $s \restriction k \neq r \restriction k$, whence $a_r^i \cap a_s^j \subseteq a_r \cap a_s$ can only contain indices of sequences with length shorter than k_0 and hence must be finite.

1.15 Definition. Let $f : A \to 2^\omega$ be a function, $A \subseteq 2^\omega$. Define P_f

$$P_f := {}^{<\omega}\omega \times [\bigcup_{r\in A}(\{r\} \times f(r))]^{<\omega},$$

ordered by

$$(s,g) \le (t,h) \Leftrightarrow s \supseteq t \text{ and for all } (r,i) \in h,\ s \setminus t \cap a_r^i = \emptyset$$

The first component of a condition should be thought of as a finite approximation to the generic real; the second component contains pairs (r,i) indexing the sets a_r^i that must be avoided by any stronger approximation.

For any two reals r, s define

$$r \odot s := \{\ i \in \omega \mid s \cap a_r^i \text{ is finite }\}$$

It remains to check the forcing does what was promised and behaves nicely.

1.16 Fact. If $B = \bigcup\{p \mid \exists q(p,q) \in G\}$ for G that is P_f-generic, then for all $r \in A$, $f(r) = r \odot B$.

This fact is an immediate consequence of the following two lemmas.

1.17 Lemma. *For each $(r,i) \in 2^\omega \times \omega$ such that $i \notin f(r)$ and for each $n \in \omega$, the set $D := \{(p,h) \in P_f \mid (p \cap a_r^i) \setminus n \neq \emptyset\}$ is dense in P_f.*

Proof. Like lemma 1.12. Let (p,h) be arbitrary. As $i \notin f(r)$, certainly $(r,i) \notin h$; so $a_r^i \setminus \bigcup_{(s,j)\in h} a_s^j$ must be infinite, as these are almost disjoint subsets of ω. So, picking $k > n$ in that latter set, $(p \cup \{k\}, h)$ is a condition in D extending (p,h). □

1.18 Lemma. *For each $r \in 2^\omega$ and each $i \in f(r)$, the set $D := \{(p,h) \in P_f \mid (r,i) \in h\}$ is dense in P_f.*

Proof. For any condition (p,h) and any pair (r,i) as above, $(p, h \cup \{(r,i)\})$ is a condition extending (p,h). □

1.19 Fact. P_f is ω-centered (and thus has the *ccc*).

Proof. Again, let $f : P_f \to {}^{<\omega}\omega \cong \omega$ be the projection to the first coordinate. If (p_i, h_i), for $i \in k$, are conditions with the same first component, indeed $(p_0, \bigcup_{i\in k} h_i)$ is also a condition. □

Shooting a club through a stationary set

Any superset of a *cub* set is stationary, but not necessarily all stationary sets are obtained in this way: if $cf(\kappa) > \omega_1$, the set of ordinals below κ with uncountable cofinality is stationary; it cannot contain a *cub* subset as any such set must have points of cofinality ω. For ω_1, the situation is different: we shall show that for any stationary subset S of ω_1, you can force to add a *cub* subset of S. If S does not contain a *cub* subset, $\omega_1 - S$ is also stationary, but ceases to be in the extension - so the forcing won't preserve stationary subsets of ω_1. As we shall see, it does preserve ω_1.

1.20 Definition. Adding a club. Let S be a stationary subset of ω_1. Define

$$P := \{s \subseteq S \mid s \text{ is closed and bounded in } \kappa\ \},$$

ordered by

$$s \leq t \Leftrightarrow t = s \cap (\bigcup t).$$

In short, conditions should be thought of as closed initial segments of a *cub* subset of S to be added, ordered by end-extension.

1.21 Fact. If G is P-generic, $\bigcup G$ is a *cub* subset of S.

Proof. As, for any $\xi < \kappa$, the set $D_\xi := \{s \in P \mid \xi < sup(s)\}$ is dense, $\bigcup G$ is unbounded in ω_1. If γ is a limit point of $\bigcup G$, take some $s \in G$ such that $sup(s) > \gamma$. As $(\bigcup G) \cap sup(s) = s$, and s is closed, it must be the case that $\gamma \in \bigcup G$, so $\bigcup G$ is closed. $\square$

1.22 Fact. P is ω_1-distributive.

Proof. Why is P not σ-closed? Because if we let γ be a limit point of S that is not contained in S, we can choose a sequence of conditions such that any candidate for a condition containing all of them as subsets must either fail to be closed or must contain $\gamma \notin S$.

With this in mind, given a sequence $(D_n)_{n<\omega}$ of dense subsets of P and a condition q_0, we shall construct a descending chain of conditions $(p_n)_{n<\omega}$ below q_0 with sufficient care so that at the limit, we can find a condition extending what has previously been chosen.

To this end, we build a chain of models $(M_\xi)_{\xi\in\omega_1}$: for successor stages $\eta = \xi + 1$ for $\xi < \omega_1$, choose M_η s.t.

$$\langle M_\eta, p_0, P, (D_n)_{n\in\omega}\rangle \prec \langle H_{\omega_1}, p_0, P, (D_n)_{n\in\omega}\rangle, \tag{4}$$
$$\text{and } \xi \cup M_\xi \subseteq M_\eta.$$

At limit stages $\eta \in \omega_1 \cap Lim$, take unions:

$$M_\lambda = \textstyle\bigcup_{\xi<\lambda} M_\xi$$

Thus, (4) will hold for all $\eta < \omega_1$. Note that by construction the set

$$C := \{\ \bigcup(M_\xi \cap On) \quad | \quad \xi < \omega_1\}$$

is *cub* in ω_1. So we can choose $\gamma \in S\cap C$. Let δ be such that $\bigcup(M_\delta \cap On) = \gamma$. Then, for each $n \in \omega$ we can choose $\gamma_n \in M_\delta$ such that $\bigcup_{n\in\omega} \gamma_n = \gamma$. Now by (4), we have that

$$M_\delta \vDash \text{“}\forall n \in \omega \quad D_n \text{ is dense in } P\text{”},$$

and for each $n \in \omega$,

$$M_\delta \vDash \text{“}\{p \in P \mid sup(p) > \gamma_n\} \text{ is dense in } P\text{”}.$$

So we can choose a sequence of conditions, starting with p_0, such that for each $n \in \omega - \{0\}$,

$$\begin{gathered} p_{n+1} \leq p_n, \\ p_{n+1} \in D_n \cap M_\delta, \text{ and} \\ sup(p_{n+1}) > \gamma_n. \end{gathered}$$

Finally, we set $p_\omega := (\bigcup_{n\in\omega} p_n) \cup \{\gamma\}$. This p_ω is a closed and thus a condition, and clearly $p_\omega \in \bigcap_{n\in\omega} D_n$, $p_\omega \leq p_0$. $\square$

1.23 Corollary. *P does not collapse ω_1.*

Collapsing orders

1.24 Definition. For $S \subseteq On$, γ regular, define

$$\begin{aligned} Coll(\gamma, S) := \{f \mid \quad & f \text{ is a function, } dom(f) \in [S \times \gamma]^{<\gamma} \text{ and} \\ & \forall (\xi, \eta) \in S \times \gamma \quad f(\xi, \eta) < \xi \quad \}, \end{aligned}$$

ordered by inclusion. $Coll(\gamma, \{\kappa\})$ is usually called the *Lévy Collapse of κ onto γ*, while $Coll(\gamma, \kappa)$ is called the *gentle Lévy Collapse of κ (onto γ^+)*.

1.25 Fact.

1. $\Vdash_{Coll(\gamma,S)} \forall \xi \in \check{S} \quad \xi \cong \gamma$
2. $Coll(\gamma, S)$ is γ-closed.
3. If $\kappa^{<\gamma} = \kappa$, $Coll(\gamma, \{\kappa\})$ has the κ^+-cc
4. Let κ be regular and greater than γ, and assume $\forall \xi < \kappa$, $\xi^{<\gamma} < \kappa$. Then $Coll(\gamma, \kappa)$ has the κ-cc

Proof.

1. Obviously, for any $\xi \in S$ and any $\chi \in \xi$, the set of conditions q where for some $\eta \in \gamma$, $p(\xi, \eta) = \chi$ is dense. Thus, if G is the generic, $F := \bigcup G$ is a function $F : S \times \gamma \to S$ such that for each $\xi \in S$, the function $\eta \mapsto F(\xi, \eta)$ is a surjection from γ to ξ.
2. Clear, as the union of less than γ many functions of size less than γ has itself size less than γ by regularity.
3. Clear, as $Coll(\gamma, \{\kappa\}) \subseteq [\{\kappa\} \times \gamma \times \kappa]^{<\gamma} \cong \kappa^{<\gamma}$.

4. By way of contradiction, let A be an antichain of $Coll(\gamma, \kappa)$ of size κ. The unwieldy hypothesis that $\forall \xi < \kappa \quad \xi^{<\gamma} < \kappa$ (and regularity of κ) allows us to use the Δ-System-Lemma for $\{dom(p)|p \in A\}$: we can find a root $r \in [\kappa]^{<\gamma}$ and $A' \subseteq A$ of size κ such that for any two distinct $p, q \in A'$, $dom(p) \cap dom(q) = r$. But as r must be bounded in κ, say, by κ_0, there are only $\kappa_0^{<\gamma} < \kappa$ possibilities for $p \restriction r$, $p \in A'$. Thus (once more by regularity of κ) κ many $p \in A'$ agree on r and thus, as their domains are disjoint outside r, they must be be compatible, in contradiction to the fact that A is an antichain.

□

A simple proof of the following simple fact is strangely absent from the usual texts on forcing, but one is included in [Kan97, p. 129]. More general theorems can be proved along similar lines, both about the Lévy-Collapse (see [Jec78, p. 280]) and about weak homogeneity (see [Jec78, p. 270]). The class of *standard names* for P is defined by induction: a P-name q is a standard name exactly if for all x, y s.t. $(x, y) \in q$, x is a standard name and $y = 1_P$, the maximal element of P. Every set a in the ground model has a standard name $\check{a}$, and for any P-generic G, $\check{a}[G] = a$.

1.26 Fact. If P is *weakly homogeneous*, that is, if for any $p, q \in P$ there is an automorphism $e : P \to P$ such that $i(p) \| q$, then for any formula ϕ of the forcing language mentioning only standard names, and for any $p \in P$,

$$p \Vdash_P \phi \iff \Vdash_P \phi.$$

1.27 Fact. The Lévy-Collapse is weakly homogeneous.

Proof. Let $p, q \in Coll(\lambda, S)$. We need to find a map e which is an automorphism of this collapsing order such that $e(p)$ and q are compatible. Pick any bijection $f : \lambda \to \lambda$ such that for all $(\chi, \xi) \in dom(p)$ and all $chi' \in S$, $(\chi', f(\xi)) \notin dom(q)$ (this is possible as $dom(p), dom(q) \lesssim \lambda$). This induces an automorphism e: we define $e(p) : \lambda \times S \to \bigcup S$ to be such that

$$e(p)(\chi, \xi) = p(\chi, f(\xi))$$

As e can be viewed as taking images under the bijection $id \times f \times id$ on $S \times \lambda \times \bigcup S$, e clearly preserves $\subseteq$ on $\mathcal{P}(S \times \lambda \times \bigcup S))$. Clearly, $e : Coll(\lambda, S) \to Coll(\lambda, S)$ is an automorphism of $Coll(\lambda, S)$ and $dom(e(p)) \cap dom(q) = \emptyset$. □

The Lévy-Collapse is very easily decomposable as a product, and the projections and complete embeddings take a very simple form:

1.28 Fact. Let $S = R \cup T$ and $R \cap T = \emptyset$. Then the map $i : \langle p, q \rangle \mapsto p \cup q$ is an isomorphism from $Coll(\lambda, R) \times Coll(\lambda, T)$ onto $Coll(\lambda, S)$ and $Coll(\lambda, T)^V = Coll(\lambda, T)^{V[G]}$, for any G that is $Coll(\lambda, S)$-generic over V.

Proof. That the mapping i is a bijection and order-preserving is obvious. The second fact holds since by λ-closedness of the first Lévy-Collapse, no new subsets of $S \times \lambda \times \bigcup S$ of size less than λ are added by G. $\square$

One of properties of the Lévy-Collapse, its so-called universality, can be loosely described thus: The Lévy-Collapse forces every upwards absolute formula that can be forced by a comparatively small forcing, or, in other words, the Lévy-Collapse adds a generic for every small forcing.

1.29 Fact. Let P be a separative p.o. and α uncountable such that $|P| \leq \alpha$ but $\Vdash_P \omega \cong \alpha$. Then there is a dense subset of $Coll(\omega, \{\alpha\})$ which can be densely embedded into P.

Proof. First, observe that below every condition, we can find an antichain of size α. Otherwise, if there were some $q \in P$ such that the α-cc holds below q, then for some regular $\beta \leq \alpha$, even the β-cc holds below q. (We tacitly use a well-known fact about the chain condition see [Jec78, p. 157]. It would, on the other hand, be of no harm to prove the present fact only for regular α.) Let $\dot{f}$ be a P-name s.t. $\Vdash_P \dot{f} : \check{\omega} \twoheadrightarrow \check{\beta}$. The set of conditions below p which decide $\dot{f}(\check{n})$ with different value is clearly an antichain; if for each n, this chain is small, i.e. the set $A_n := \{\xi < \alpha | \exists q \leq p \text{ s.t. } q \Vdash \dot{f} = \check{\xi}\}$ is bounded in β, by regularity of β, $\dot{f}$ will be forced by q to be bounded in β, a contradiction.

The embedding - call it i - will be defined on ${}^{<\omega}\alpha$, which is of course dense in $Coll(\omega, \{\alpha\})$. Fix a name $\dot{g}$ such that $\Vdash_P \dot{g} : \check{\omega} \twoheadrightarrow \dot{G}$. Now we define i inductively on the length of conditions. Of course, the empty sequence $1_{Coll(\omega,\{\alpha\})}$ can be mapped simply to 1_P. Now let's assume we have defined i on ${}^n\alpha$. For each $q \in {}^n\alpha$, let $A^{i(q)} := \{a^{i(q)}_\xi | \xi < \alpha\}$ be a maximal pairwise incompatible subset of $\{p \in P | p \leq i(q)\}$, of size α, and such that each element of $A^{i(q)}$ decide the value of $\dot{g}(\check{n})$ (enlarge an antichain below $i(q)$ until it is maximal and refine it to decide the values of $\dot{g}$). Then for $p \in {}^{n+1}\alpha$, let $i(p) := a^{i(p\restriction n)}_{p(n)}$.

It is obvious from the construction that i is injective and i and i^{-1} are order-preserving. It thus remains to show that $i''({}^{<\omega}\alpha)$ is dense in P. So take any $p \in P$. As $p \Vdash \check{p} \in \dot{G}$, for some n and some $p' \leq p$, $p' \Vdash \dot{g}(\check{n}) = \check{p}$. Of course, p' must be compatible to some element a of $A^{\emptyset}$; then again, to some element of A^a, as this is a maximal antichain below a; going on like this, p' has to be compatible with one of the elements, say a', of $A^{\bar{a}}$, for some $\bar{a}$. As

a' itself also decides the value of $\dot{g}(\check{n})$, it must agree with p' on this value, whence $a' \Vdash \check{p} \in \dot{G}$. But then, by separativity of P, even $a' \leq p$. $\square$

1.30 Corollary. Universality. *Let P be a p.o. and α regular such that $|P| < \alpha$. Then there exists a p.o. Q and a complete embedding $i : P \to Q$ where $Q \sim Coll(\omega, \{\alpha\})$, i.e. Q is equivalent to the Lévy-Collapse of α onto ω.*

Proof. Consider $Q := P \times Coll(\omega, \{\alpha\})$. By the previous fact, there exists a dense embedding $d : {}^{<\omega}\alpha \to Q$, and ${}^{<\omega}\alpha$ is dense in $Coll(\omega, \{\alpha\})$. So obviously $Q \sim Coll(\omega, \{\alpha\})$, and $p_0 : P \to Q$, the canonical projection to the first coordinate, is a complete embedding. $\square$

1.3 Reals

To make notation easier, for this section, let variables m, n always denote natural numbers, and variables s, w elements of $Seq := {}^{<\omega}\omega$. By reals we usually mean members of 2^ω, and we will issue a warning before we switch to ${}^\omega\omega$. Fix a bijection $\Gamma : \omega^2 \longrightarrow \omega$, which is arithmetical, and an arithmetical enumeration of ${}^{<\omega}\omega$, $(s_n)_{n\in\omega}$. Thus we are allowed to talk about finite sequences of natural numbers as if they were natural numbers. We denote the length of an ω-sequence s by $lh(s)$.

We shall need a normal form for Σ^1_n relations, which we can easily state as a Definition 1.31. Π^1_n, Δ^1_n and boldface and lightface versions of the hierarchy are then defined as usual (see, e.g., [Jec78, p. 500] for a survey of descriptive set theory and [Jec78, p. 509ff.] for definitions compatible with the one given here). By $\Delta^1_0(a_1, \ldots, a_l)$ or *arithmetical in* $a_1, \ldots, a_l$, we mean definable in the model $\langle \omega, a_1, \ldots, a_l, \in \rangle$. When we talk about trees, for simplicity of notation, consider them to be ordered by reverse inclusion. Thus our trees will grow downward, as they do in Israel. An infinite branch through a tree is an infinite descending chain of nodes. By a *ranking function*, we mean an order-preserving function into the ordinals always taking the least possible value.

1.31 Definition. A k-ary relation on ${}^\omega\omega$ is $\Sigma^1_n(a_1, \ldots, a_l)$ exactly if it can be written in the following form:

$$(x_1, \ldots, x_k) \in A \iff \underbrace{\exists\, z_1 \in \omega^\omega \ \ldots Q z_n \in \omega^\omega}_{n \text{ blocks of quantifiers}} \ B(x_1, \ldots, x_k, z_1, \ldots, z_n),$$

where each block consists exclusively of quantifiers of one type, either $\forall$ or $\exists$ (we have written Q for the last quantifier, $\forall$ for even n, $\exists$ for odd n) and where $B(x, z_1, \dots, z_n)$ is a relation arithmetical in $a_1, \dots, a_l$. Moreover for $n > 0$, we can demand that B be equivalent to the following statement if n is even and equivalent to the negation if n is odd:

$$\exists m \in \omega \quad (x_1 \restriction m, \dots, x_k \restriction m, z_1 \restriction m, \dots, z_n \restriction m) \notin T,$$

for a tree T on Seq^{k+n}, arithmetical in $a_1, \dots, a_l$.

Hereditarily countable sets and reals

1.32 Definition. We say that $r \in 2^\omega$ *codes* $x \in HC$ if, and only if, the relation $E_r := \{(m,n) \in \omega \times \omega \mid \Gamma(m,n) \in r\}$ is well-founded, extensional and the Mostowski-collapse of E_r is just $TC(\ \{x\}\ )$. We say that *r codes x via $g : TC(\ \{x\}\ ) \longrightarrow \omega$* if all of the above holds and g is the inverse of the collapsing map. We write $r_0 \overset{r}{\cong} r_1$, if E_{r_0} and E_{r_1} are extensional and $E_r \subseteq \omega \times \omega$ is the graph of a partial function f_r, s.t. $\forall m, n \in dom(f_r)\, [mE_{r_0}n \iff f_r(m)E_{r_1}f_r(n)]$. Denote the m with no successors in E_r by $top(r)$ (if r codes x via g, $g(x) = top(r)$). Let $field(r) := dom(E_r) \cup range(E_r)$, and $r \sqcup s := \{\Gamma(2^n, 2^m) | nE_rm\} \cup \{\Gamma(3^n, 3^m) | nE_sm\}$. For $m \in field(r)$, let

$$tc_r(m) := \{n \mid \exists k\, \exists s \in \omega^{k+1} \wedge s(0) = n \wedge s(k) = m \wedge \forall l < k\ s(l)E_rs(l+1)\}$$

Observe all the notions presented are arithmetical (with real parameters). Of course, for any set $x \in HC$, we can find a real r_x coding x. Just choose some $g : TC(\{x\}) \xrightarrow{1-1} \omega$ and let $r_x := \Gamma'' E_x$, where $\langle TC(\{x\}), \in\rangle \overset{g}{\cong} \langle \omega, E_x\rangle$.

Assume r_0, r_1 code p_0, p_1. "$p_0 \in p_1$" is $\Delta^1_1(r_0, r_1)$:

$$\exists r\, r_0 \overset{r}{\cong} r_1 \wedge field(r_0) \subseteq dom(f_r) \wedge f_r(top(r_0))E_{r_1}top(r_1)$$

$$\forall s \forall t\, [r_0 \sqcup r_1 \overset{s}{\cong} t \wedge field(r_0 \sqcup r_1) = dom(f_s) \wedge ran(f_s) = field(t)] \Rightarrow$$
$$\Rightarrow f_s(2^{top(r_0)})E_t f_s(3^{top(r_1)})$$

"$p_0 = p_1$" is also $\Delta^1_1(r_0, r_1)$:

$$\exists r\, r_0 \overset{r}{\cong} r_1 \wedge field(r_0) \subseteq dom(f_r) \wedge f_r(top(r_0)) = top(r_1)$$

$$\forall s \forall t\, [r_0 \sqcup r_1 \overset{s}{\cong} t \wedge field(r_0 \sqcup r_1) = dom(f_s) \wedge ran(f_s) = field(t)] \Rightarrow$$
$$\Rightarrow f_s(2^{top(r_0)}) = f_s(3^{top(r_1)})$$

Moreover, for variables x_i ranging over $TC(\ \{p_i\}\ )$, respectively ($i \in 2$), we can replace them by variables ranging over ω and express "$x_0 \in x_1$" and "$x_0 = x_1$" using the formulas above, with x_i instead of $top(r_i)$ and $tc_{r_0}(x_0)$ instead of $field(r_0)$.

1.33 Fact. Let $\Psi(x_0, \ldots, x_k)$ be a statement in the language of set theory, $(p_0, \ldots, p_k) \in HC$. Then there exists a formula $\Phi(x_0, \ldots, x_k)$ s.t. for any reals $r_0, \ldots r_k$ coding $p_0, \ldots, p_k$,

1. $(\ \langle HC, \in\rangle \vDash \Psi(p_1, \ldots, p_k)\ ) \iff \Phi(r_1, \ldots, r_k)$
2. If $\Psi(p_1, \ldots, p_k)$ is $\mathbf{\Sigma_n}$, $\Phi(r_1, \ldots, r_k)$ is $\mathbf{\Sigma^1_{n+1}}$
   If $\Psi(p_1, \ldots, p_k)$ is $\mathbf{\Pi_n}$, $\Phi(r_1, \ldots, r_k)$ is $\mathbf{\Pi^1_{n+1}}$.

Proof. Let $\Psi = \Psi(x_0, \ldots, x_k)$, $p_0, \ldots, p_k$, $r_0, \ldots r_k$ be given as above. The proof goes by induction on formula complexity. For the $\mathbf{\Delta_0}$ case, replace atomic formulas in the way indicated in the preceding discussion. From what has been said there, 1. is immediate. By shifting quantifiers to the front, we see we have a $\mathbf{\Delta^1_1}$ statement. Now assume Ψ is $\mathbf{\Sigma_{n+1}}$ and we can already convert $\mathbf{\Pi_n}$ statements. For simplicity, assume $\Psi(p) = \exists x \Theta(x, p)$, where $\Theta(x, p)$ is $\mathbf{\Pi_n}$. We want to express

$$\exists x \in HC \quad HC \vDash \Theta(x, p). \tag{5}$$

By induction, $HC \vDash \Theta(x, p)$ converts to a $\mathbf{\Pi^1_{n+1}}$ formula. We see that (5) is equivalent to saying "$\exists R$ s.t. R codes an extensional, well-founded relation and $\bar{\Theta}(R, r_p)$ holds", where $\bar{\Theta}$ is $\mathbf{\Pi^1_{n+1}}$.

"$R \in 2^\omega$ codes a well founded relation" $\iff$ "there is no $R' \in 2^\omega$ such that $f(m, n) \in R'$ just if n is the m-th element of an infinite branch through R", i.e., this is a $\mathbf{\Pi^1_1}$-property of R. So we get that (5) is equivalent to a statement of the form

$$\exists R \in 2^\omega \quad \forall R' \in 2^\omega \quad \neg\Upsilon(R', R) \ \wedge\ \bar{\Theta}(R, r_p) \tag{6}$$

where $\Upsilon(R', R)$ is a $\mathbf{\Delta^1_0}$ statement asserting that R' codes an infinite branch through R, and thus the whole of (6) is $\mathbf{\Sigma^1_{n+2}}$. This completes the inductive step for the $\mathbf{\Sigma^1_{n+1}}$-case, and the $\mathbf{\Pi^1_{n+1}}$-case works just the same, mutatis mutandis. $\square$

Levy-Shoenfield absoluteness theorem

This paper deals with $\mathbf{\Sigma^1_3}$-absoluteness between a ground model and the forcing extension. What about $\mathbf{\Sigma^1_2}$-absoluteness? In this case, Theorem 1.34, called Levy-Shoenfield Absoluteness Theorem, shows absoluteness holds not only between a ground model and its extension, but for a large class of transitive models.

For this section, reals will be elements of ${}^\omega\omega$, rather than elements of 2^ω.

1.34 Theorem. *Let A be a $\Sigma^1_2(a)$-predicate, for a real a, and let M be a transitive model of ZF^-, $a \in M$, $\omega_1 \subseteq M$. Then*

$$A^M = A \cap M$$

Proof. Let U a model of ZF^- (possibly $U = V$) s.t. $a \in U$, $\omega_1 \subseteq U$ and work in U. For a $\Sigma^1_2(a)$-predicate A the following equivalences clearly hold, with κ denoting the ω_1 of V,

$$\begin{aligned} x \in A \quad &\iff \exists\, y \in \omega^\omega\, \forall\, z \in \omega^\omega\, \exists n \in \omega \quad (x \restriction n, y \restriction n, z \restriction n) \notin T \\ &\iff \exists\, y \quad \text{s.t. } T(x,y) \text{ is well founded.} \\ &\iff \exists\, y\, \exists\, f : Seq \longrightarrow \kappa \text{ s.t. } f \text{ is order-preserving on } T(x,y). \end{aligned}$$

Here T is arithmetical in a parameter a and

$$T(x,y) := \{w \in Seq \mid (x \restriction lh(w), y \restriction lh(w), w) \in T\}.$$

The last equivalence holds as the ZF^--model U can build a rank function for the countable tree $T(x,y)$ just if it is well-founded.

Now we define another tree $\bar{T}$ in U (the nodes correspond to initial segments of a ranking function):

$$\begin{aligned} \bar{T} := \{(u,v,r) \in Seq \times Seq \times {}^{<\omega}\kappa \mid \quad & lh(u) = lh(v) = lh(r) \text{ and} \\ & \bar{r} : Seq \xrightarrow{par} \kappa \text{ is} \\ & \text{order-preserving on } T(u,v).\} \end{aligned}$$

where $\bar{r}$ denotes the partial function $\bar{r}(s_n) = r(n)$ ($(s_k)_{k\in\omega}$ is your favorite canonical enumeration of sequences of natural numbers), and $T(u,v) := \{w \in {}^{lh(u)}\omega \mid \quad (u \restriction lh(w), v \restriction lh(w), w) \in T\}$. This allows us to write

$$x \in A \iff \bar{T}(x) \text{ not well-founded,}$$

where of course $\bar{T}(x) := \{(v,r) \mid (x \restriction lh(t), v, r) \in \bar{T}\}$. Now we leave U. Observe that, as $\bar{T}$ is $\Delta_1(a)$, $\bar{T}^U = \bar{T}^V$. The usual way of showing absoluteness of well-foundedness (via a rank function on $\bar{T}(x)$) easily yields that for M as in the hypotheses of the theorem, and $x \in M$,

$$x \in A \iff M \models x \in A$$

□

1.4 Large cardinals

Zero sharp and the Covering Lemma

$0^\sharp$ is a set of natural numbers satisfying a certain, absolute syntactical property. $0^\sharp$ is not in L; in fact, we may regard it as a truth predicate for L. Its existence is equivalent to the existence of a *cub* class of ordinals, containing all uncountable V-cardinals, which are indiscernible in L and whose Skolem hull in L is all of L. An introduction to $0^\sharp$ and indiscernibles can be found in [Jec78, sec. 30] or [Dev84, V]. The next two facts can serve as a black box for our purposes. For a proof, see [DJ75].

1.35 Fact. If $0^\sharp$ exists, $V \neq L$ and for any two uncountable V-cardinals $\alpha < \beta$, $L_\alpha \prec L_\beta$.

1.36 Fact. Covering Lemma. If $0^\sharp$ does not exist, $\forall\, X \subseteq L\, \exists\, Y \in L$ s.t. $|Y| = |X| + \omega_1 \wedge X \subseteq Y$

We shall use the following consequences of the covering lemma. For sake of completeness, we include the proof.

1.37 Fact. Assume $0^\sharp$ does not exist. Then

1. If $2^{cf\kappa} \leq \kappa^+$, then $\kappa^{cf\kappa} = \kappa^+$ (i.e., the Singular Cardinal Hypothesis holds, implying GCH at strong limit singular cardinals).

2. If $\omega < cf\lambda < |\lambda|$, $(cf\lambda < \lambda)^L$.

3. If κ is a singular cardinal, then $(\kappa^+)^L = \kappa^+$.

Proof. 1. For each $s \in [\kappa]^{cf\kappa}$, let $s \subset Y_s \cong cf\kappa \cdot \omega_1$, $Y_s \in L \cap \mathcal{P}(\kappa)$. We have

$$[\kappa]^{cf\kappa} = \bigcup\{[Y_s]^{cf\kappa} \mid Y_s \in L_{\kappa^+}, s \in [\kappa]^{cf\kappa}\}, \text{ whence}$$

$$[\kappa]^{cf\kappa} \lesssim \kappa^+ \cdot (cf\kappa \cdot \omega_1)^{cf\kappa} = \kappa^+ \cdot 2^{cf\kappa} = \kappa^+$$

2. Let $C \overset{cof}{\subset} \lambda$, $otp\, C = cf\ \lambda$. Take $C \subseteq Y \in L$, $Y \cong cf\ \lambda$. Then $otp\, Y < (cf\ \lambda)^+ \leq |\lambda| \leq \lambda$, but the order type is absolute between L and V.

3. By 2, any ordinal between κ and κ^+ must have cofinality less than itself in L. In particular, it cannot be a regular cardinal in L.

$\square$

Some small large cardinals

The consistency proofs in this paper will deal with *reflecting*, *remarkable* and with *lightface* Σ^1_2-*indescribable* cardinals (which will be introduced and treated in section 5.1). We would like to give convenient upper and lower bounds for the strength of these notions; any definitions we omit here can be found in [Kan97].

1.38 Definition. We say $\kappa \in Reg$ is *reflecting* if, and only if, for all formula $\phi(x)$ and $\forall\, p \in H_\kappa$, whenever $\exists\, \Theta\; H_\Theta \models \phi(p)$, $\exists\, \delta < \kappa\; H_\delta \models \phi(p)$.

1.39 Fact. Let κ be a regular cardinal. Then κ is reflecting $\iff V_\kappa \prec_{\Sigma_2} V$.

Proof. First assume κ is reflecting. Observe that $H_\kappa = V_\kappa$: κ is inaccessible, as $\forall x \in H_\kappa\, \exists \delta < \kappa$ s.t. $H_\delta \models \exists \mathcal{P}(x)$, and $\mathcal{P}(x)^{H_\delta} = \mathcal{P}(x) \cap H_\delta = \mathcal{P}(x)$. κ inaccessible implies $H_\kappa = V_\kappa$, as in that case, H_κ is closed under power set and small unions. As κ is an uncountable regular cardinal, $H_\kappa \prec_{\Sigma_1} V$, and so upwards absoluteness of Σ_2-statements is immediate. Now if $V \models \exists x \phi(x)$ (where $\phi(x)$ is Π_1 with parameters in H_κ), there is $x_0 \in H_\delta$, $\delta < \kappa$ s.t. $H_\delta \models \phi(x_0)$, and $\phi(x_0)$ is absolute between H_δ and H_κ.

Now, conversely, suppose $V_\kappa \prec_{\Sigma_2} V$. First of all, again κ must be inaccessible: asserting existence of a surjection from an ordinal to $\mathcal{P}(x)$ is Σ_2, while $\mathcal{P}(x) \in V_\kappa$ for $x \in V_\kappa$ (as κ is limit). So again $H_\kappa = V_\kappa$. Secondly, suppose $H_\theta \models \phi$, allowing parameters in H_κ. "$y = H_\delta$" is Π_1 (and hence absolute between H_κ and V) so $\exists \delta\; H_\delta \models \phi$ is Σ_2, and thus is reflected by $H_\kappa \prec_{\Sigma_2} V$. $\square$

1.40 Fact. If both a reflecting and a Mahlo cardinal exist, then the least Mahlo is strictly below the least reflecting, which itself is not Mahlo. Existence of a Mahlo implies consistency of a stationary class of reflecting cardinals.

Proof. Let θ be the least Mahlo and κ the least reflecting. If $\theta > \kappa$, as $H_{\theta^+} \models$"$\exists$ a Mahlo", some H_η, with regular $\eta < \kappa$ reflects this; as being Mahlo is absolute between H_η and V, there is a Mahlo below κ, contradiction. Now assume κ itself is Mahlo. For any ordinal $\xi < \kappa$, let $f(\xi)$ be least such that for all $p \in H_{\xi^+}$ and all formulas $\phi(x)$, if there is a regular δ s.t. $H_\delta \models \phi(p)$, then there is such δ below $f(\xi)$. By the reflection property and inaccessibility of κ, f maps κ into κ; as κ is Mahlo, there must be a regular fixed point of f below κ, but such a point must be a reflecting cardinal itself. If θ is any Mahlo (not reflecting), look at a similar f such that its regular fixed points are reflecting in H_θ; H_θ is a model of ZFC with stationarily many reflecting cardinals. $\square$

Let us call a cardinal Σ_k-Mahlo (resp. Π_k-Mahlo) if every *cub* subset of κ with a Σ_k (resp. Π_k) definition (with a parameter from V_κ) contains an inaccessible cardinal.

1.41 Fact. If κ is reflecting, κ is Σ_2-Mahlo.

Proof. Assume $\xi \in C \iff \phi(\xi)$, where $\phi(x)$ is Σ_2. As $V_\kappa \prec_{\Sigma_2} V$, we have $V_\kappa \models \forall\xi\exists\bar{\xi} > \xi\phi(\bar{\xi})$, so for some inaccessible $\eta < \kappa$, V_η believes the same thing, and by upwards absoluteness of Σ_2 statements for V_η, C is unbounded in η, whence $\eta \in C$. $\square$

1.42 Definition. [Sch00b] κ is called *θ-remarkable* for a regular cardinal θ, if, and only if, there exist countable transitive models M, N, together with elementary embeddings $\pi : M \to_{\Sigma_\omega} H_\theta$, $\sigma : M \to_{\Sigma_\omega} N$ such that for $\bar{\kappa} := \pi^{-1}(\kappa)$, $\bar{\theta} := M \cap On$,

- $crit(\sigma) = \bar{\kappa}$,
- $\sigma(\bar{\kappa}) > \bar{\theta}$
- $\bar{\theta} \in Reg^N$, $M \in N$ and $N \models$“$M = H_{\bar{\theta}}$”.

κ is *remarkable* if, and only if κ is θ-remarkable for all regular $\theta > \kappa$.

1.43 Fact. If κ is remarkable, κ is reflecting.

Proof. Assume κ is remarkable and $H_\theta \models \phi(p)$, where $p \in H_\kappa$ and $\theta > \kappa$ regular. Take N, M countable transitive, together with elementary embeddings $\pi : M \to_{\Sigma_\omega} H_\theta$, $\sigma : M \to_{\Sigma_\omega} N$, as in the definition of remarkability. Let $\bar{\kappa} := \pi^{-1}(\kappa)$. By elementarity,

$$M \models \phi(\pi^{-1}(p)).$$

But $N \models M = H_{M\cap On}$, and $M \cap On < \sigma(\bar{\kappa})$ is a regular cardinal in N. Thus

$$N \models \text{“}\exists\lambda < \sigma(\bar{\kappa})\ H_\lambda \models \phi(\pi^{-1}(p))\text{”}$$

As $\pi^{-1}(p) \in {H_{\bar{\kappa}}}^M$, we have $\sigma(\pi^{-1}(p)) = \pi^{-1}(p)$. So we can pull back via σ to obtain

$$M \models \text{“}\exists\lambda < \bar{\kappa}\ H_\lambda \models \phi(\pi^{-1}(p))\text{”}.$$

Once more applying π, we can see that $H_\lambda \models \phi(p)$ for some regular cardinal λ below κ. $\square$

To show relative consistency of a remarkable cardinal, we mention the *ω-Erdös* cardinal $\kappa(\omega)$, i.e. the least κ such that $\kappa \to (\omega)_2^{<\omega}$ (meaning every $f : [\kappa]^{<\omega} \to 2$ has an infinite homogeneous set). Such κ are inaccessible (by a combinatorial argument, see [Kan97, 7.15, p. 82]).

1.44 Fact. $\exists\kappa$ s.t. $\kappa \to (\omega)^{<\omega}_2$ implies the consistency of "$ZFC\wedge$ there is a remarkable cardinal".

Proof. Observe that if $\kappa \to (\omega)^{<\omega}_2$, then the same holds relativized to L: let f be a constructible coloring of finite sequences of κ. A homogeneous set of order-type ω exists if and only if $\{p \in [\kappa]^{<\omega} | p \text{ is } f\text{-homogeneous }\}$, ordered by $\supset$, has an infinite branch. As has been shown before (see the proof of 1.34), such statements are sufficiently absolute, so a homogeneous set exists in L. We work in L for the rest of the proof. Let κ be least such that there is $\langle L_\kappa, \iota_i\rangle_{i\in\omega}$, a structure with ω indiscernibles. Let $\pi : L_\gamma \to_{\Sigma_\omega} L_\kappa$ be an elementary embedding obtained by collapsing the Skolem hull of the indiscernibles with respect to L_κ (observe $\gamma < \omega_1$). Let α, β denote the pre-image under π of ι_0, ι_1. As L_γ is the Skolem hull of the pre-image of the indiscernibles, the map $\iota_k \mapsto \iota_{k+1}$ induces an elementary embedding $\sigma : L_\gamma \to_{\Sigma_\omega} L_\gamma$ such that $\sigma(\alpha) = \beta$ and (by minimality of κ) $\alpha = crit(\sigma)$ (we omit some details here; see [Kan97, sec. 9]). Note that by 1.5, ι_0 is regular in L_κ, and playing with σ and π shows ι_0 and hence every indiscernibles is inaccessible in L_κ (and thus in L). Now we show that $L_\beta \models$"α is remarkable" (it also is a model of ZFC by inaccessibility). Let θ be regular in L_β. Set $\bar\pi := \pi \restriction L_\theta$, $\bar\sigma := \sigma \restriction L_\theta$. It remains to observe that as β is inaccessible in L_γ, θ is also regular in L_γ and thus in $L_{\sigma(\theta)}$. We have:

$$\begin{gathered}\exists\bar\theta, \tilde\theta < \omega_1 \quad \exists\bar\pi : L_{\bar\theta} \to L_{\pi(\theta)}, \bar\sigma : L_{\bar\theta} \to L_{\tilde\theta} \text{ s.t.}\\ crit(\bar\sigma) = \bar\pi^{-1}(\pi(\alpha)),\ \bar\sigma(crit(\bar\sigma)) > \bar\theta \text{ and } \bar\theta \text{ is regular in } L_{\tilde\theta}\end{gathered} \tag{7}$$

By inaccessibility of $\pi(\beta)$, $L_{\pi(\beta)} \prec_{\Sigma_1} L$, so (7) holds in $L_{\pi(\beta)}$; pulling back via π shows that α is θ-remarkable in L_β; but θ was arbitrary. □

1.45 Fact. Every Silver indiscernible is remarkable in L.

Proof. Let $(\iota_k)_{k\in\omega+1}$ be the first $\omega+1$ indiscernibles; it suffices to show ι_0 is remarkable in L_{ι_1}. Let the real r consist of the Gödel-numbers $\{\sharp\phi(\iota_0,\iota_1) | L_{\iota_2} \models \phi(\iota_0,\iota_1)\}$. Then $r \in L$. The following sentence is $\Sigma_1(r)$: there exists a well-founded model M with ω many indiscernibles with height a limit ordinal such that the theory of M includes r. Therefore, by Levy-Shoenfield absoluteness 1.34, this is true in L. The previous proof shows that the Gödel-number of the formula saying "ι_0 is remarkable in ι_1" is in r and we are done. □

2 Equiconsistency for set forcing

2.1 Theorem. *[FMW92] The following are equiconsistent, modulo ZFC:*

1. *$\mathbf{\Sigma^1_3}$-absoluteness for all set forcing holds*

2. *there exists a reflecting cardinal*

Proof. First, assume $\mathbf{\Sigma^1_3}$-absoluteness for all set forcing holds. We will show that $L \vDash$ "κ is reflecting", for $\kappa = \omega_1^V$, using the characterization of reflectingness given in Fact 1.39 (on p. 19).

Firstly, κ is a limit cardinal in L: Else, assume $L \vDash$"$\kappa = \lambda^+ \wedge \lambda \in Card$". The following formula $\psi(\lambda)$ expresses that there exists another countable L-cardinal above λ:

$$\exists x < \omega_1 \quad x > \lambda \quad \wedge \quad \forall \alpha < \omega_1 \quad L_\alpha \vDash x \in Card$$

As both "$y = L_x''$ and $y \vDash \phi(x)$ (for any formula ϕ) are $\mathbf{\Delta}_1$, $\psi(\lambda)$ is Σ_2 over HC, in the parameter $\lambda \in HC$. So by fact 1.33 we can find a real r_λ coding λ and a formula $\bar{\psi}(r_\lambda)$ which is equivalent to $\psi(\lambda)$ and Σ^1_3 in r_λ.

Now force with $Col(\omega, \{\kappa\})$: In the extension, $\kappa < \omega_1$ holds (by fact 1.25) so $\psi(\lambda)$ holds. As this has been shown to be equivalent to a $\Sigma^1_3(r_\lambda)$ formula, it must also hold in the ground model, by $\mathbf{\Sigma^1_3}$-absoluteness. If $\alpha < \omega_1$ and is not an L-cardinal, this must be reflected by some L_γ, for $\gamma < \omega_1$. So any α witnessing $\psi(\lambda)$ really has to be an L-cardinal, $\lambda < \alpha < \omega_1$, contradicting $(\kappa = \lambda^+)^L$.

Secondly, we show $L_\kappa \prec_{\Sigma_2} L$. So let $p \in H_\kappa^L = L_\kappa$, $L \vDash \exists x \phi(x, p)$, where $\phi(x)$ is Π_1 with parameter x. Using reflection, choose some regular δ such that $L_\delta \vDash \exists x \phi(x, p)$; Now force with $Col(\omega, \{\delta\})$. In the extension, $\delta \cong \omega$, and so the formula following $\psi(p)$ holds:

$$\exists \chi < \omega_1 \quad \forall \alpha < \omega_1 \quad L_\alpha \vDash \chi \in Card \quad \wedge \quad p \in L_\chi \quad \wedge \quad L_\chi \vDash \phi \exists (x, p)$$

Just like above, this formula can be seen to be Σ_2 over HC in its parameter p, and thus equivalent to a formula $\bar{\psi}(r_p)$ which is Σ^1_3 in a parameter r_p coding p. Interestingly we have not yet used the fact that ϕ itself has bounded complexity. As before, we can now argue that $\psi(p)$ must also hold in the ground model and that there must exist an L-cardinal $\bar{\delta} < \kappa = \omega_1^V$ such that $L_{\bar{\delta}} \vDash \exists x \phi(x, p)$. But as $\bar{\delta}$ is an L-cardinal, $L_{\bar{\delta}} = H_{\bar{\delta}}^L \prec_{\Sigma_1} L_\kappa$, so Σ_2 formulas with parameters from the smaller model are upwards absolute between these two models, whence $L_\kappa \vDash \exists x \phi(x, p)$. This completes the proof that $L \vDash$ "κ is reflecting".

Now, for the other direction, assume κ is reflecting. Of course κ is inaccessible. Let G be generic for $Col(\omega, \kappa)$ over V. Then by fact 1.25, $V[G] \vDash \kappa = \omega_1$. We show $V[G] \vDash \Sigma^1_3(set-forcing)$.

So let $\phi(r)$ be some Σ^1_3 statement with real parameter $r \in V[G]$ and let $Q \in V[G]$ be a p.o. such that $V[G] \vDash$ " $\Vdash_Q \phi(r)$". We must show that already $V[G] \vDash \phi(r)$. Choose a nice name $\dot{r}$ for r, that is, a name of the form $\bigcup_{n\in\omega} \check{n} \times A_n$, where A_n is an antichain, and observe that the value of $\dot{r}$ is already decided in an initial segment of $Col(\omega, \kappa)$: each $A_n \lesssim \kappa$ and we can choose β regular such that $\bigcup_{n\in\omega}\{dom(q)|q \in A_n\} \subset \beta$. By fact 1.28, $G_0 := G \cap Col(\omega, \beta)$ is $Col(\omega, \beta)$-generic over V and $G_1 := G \cap Col(\omega, \kappa - \beta)$ is $Col(\omega, \kappa - \beta)$-generic over V. By weak homogeneity of the Lévy-Collapse, and since $\dot{r}[G] = r$, $V[G_0] \vDash$ " $\Vdash_{Col(\omega,\kappa-\beta)*Q} \phi(\check{r})$".

Obviously, $V[G_0] \vDash$ "$\exists P \Vdash_P \phi(\check{r})$", so in V, the following holds: there is a name $\dot{P}$ for a p.o. such that $\Vdash_{Col(\omega,\beta)}$" $\Vdash_{\dot{P}} \phi(\check{r})$". Now the fact that κ is reflecting in V can be used to show that without loss of generality, $\dot{P} \in H_\alpha$, for some $\alpha < \kappa$. Of course, for $P := \dot{P}[G_0]$, $|P| < \alpha$ in $V[G_0]$. By universality of the Lévy-Collapse (corollary 1.30), there exists $i : P \to Q$, a complete embedding into a p.o. Q that is equivalent to $Col(\omega, \{\alpha\})$.

Now we go back to $V[G]$. Of course, by fact 1.28, $J_C := G \cap Col(\omega, \{\alpha\})$ is $Col(\omega, \{\alpha\})$-generic over $V[G \cap Col(\omega, \alpha)]$, and a fortiori over $V[G_0]$. As the property of being a dense or complete embedding is absolute for transitive models, J_C can be used to construct a generic J_Q on Q (as in 1.8, via dense embeddings) and then (again as in 1.8, via i) to construct a generic (still over $V[G_0]$) for P - call it H. As $\Vdash_P \phi(\check{r})$ over $V[G_0]$, $V[G_0][H] \vDash \phi(r)$. But again by completeness of i and the existence of various dense and complete embeddings, $V[G_0][H] \subseteq V[G_0][J_Q] = V[G_0][J_C] \subseteq V[G]$, and, being Σ^1_3 in r, $\phi(r)$ is upward absolute for transitive inner models of $V[G]$ containing r, whence $V[G] \vDash \phi(r)$. This completes the proof that $V[G]$ is a model of $\boldsymbol{\Sigma}^1_3$-absoluteness for set forcing. □

3 A lower bound for ω_1-preserving set-forcing

3.1 Theorem. *[FB01] Assume $\mathbf{\Sigma^1_3}$-absoluteness for all ω_1-preserving set forcings. Then, for $\kappa = \omega_1^V$, $L \vDash$ "κ is inaccessible".*

Proof. Assume that, to the contrary, $L \vDash$"$(\omega_1)^V = \lambda^+$". Using the absoluteness assumption, we will obtain a contradiction. As $\lambda < \omega_1$, let $f : \omega \twoheadrightarrow \lambda$. Then $f \in HC$ and $\omega_1^{L[f]} = \omega_1$. Further, for each $\xi < \omega_1$, let g_ξ denote the $<_{L[f]}$-least surjection $g_\xi : \omega \twoheadrightarrow \xi$. For each $n \in \omega$, $g_\xi(n) < \xi$, so the function $h_n : \omega_1 \to \omega_1$, defined by $\xi \mapsto g_\xi(n)$ is regressiv. Hence, by Fodors Lemma , there exists a stationary subset S_n of ω_1 such that h_n is constant on S_n.

Clearly, the intersection of all the S_n cannot contain two distinct ordinals: as for all $\alpha, \beta \in S_n$, $g_\alpha(n) = g_\beta(n)$, if $\alpha, \beta \in S_n$ for all $n \in \omega$, we have $g_\alpha = b_\beta$, contradicting the definition of these functions. Of course, the intersection of even only two stationary sets needn't be stationary. But, using an ω_1-preserving forcing, we can add a *cub* subset to any one of these stationary sets; and the *cub* subsets of ω_1 form a σ-complete filter. The idea of this proof is to use $\mathbf{\Sigma^1_3}$-absoluteness to pullback such a *cub* subset with the desired property into the ground model. While forcing will usually not be able to add *cub* subsets to the S_n for all n simultaneously, these pullbacks co-exist in the ground model, leading to a contradiction.

So fix n for the moment. We force with P_{S_n}, the forcing for adding a *cub* subset of this particular S_n (see definition 1.20, p. 10). So let G be generic for P_{S_n} and work in $V[G]$, denoting $\bigcup G$, the generic *cub* subset of S_n, by C. In this model, clearly

$$\exists C \textit{ cub} \text{ in } \omega_1 \text{ s.t. } \forall \xi, \chi \in C,\ g_\xi(n) = g_\chi(n).$$

We will now force again so that in the extension, a Σ_2 strengthening the previous sentence with parameters from HC^V holds. Look at the following function $c : 2^\omega \to 2^\omega$: if $r \in 2^\omega$ codes a countable ordinal α (in the sense that it codes a relation whose transitive collapse is the $\in$-relation on α, see 1.32), define $c(r)$ to be some real that codes the countable ordinal $min(C - \alpha)$. Now we force with P_c, the almost disjoint coding for c (definition 1.15, p. 8). Let R be generic for this forcing and work in $V[G][R]$.

Now we make the following observation: if $\alpha < \omega_1$ is such that $L_\alpha[f][R]$ is a model of ZF^- and all ordinals below α are seen to be countable inside $L_\alpha[f][R]$, then $\alpha \in C$. To see this, let $\eta < \alpha$ be arbitrary. As $L_\alpha[f][R] \vDash$"$\eta \cong \omega$", we can easily find $E \in 2^\omega \cap L_\alpha[f][R]$ coding η (let f be a bijection from ω to η in that model and let E code the pullback of the relation $\in$ on η under that bijection). So as $E \odot R \in L_\alpha[f][R]$ and $L_\alpha[f][R] \vDash ZF^-$ (in fact

Δ_1-Replacement would suffice), $c(E) = tcoll(\langle \omega, E \odot R\rangle) \in L_\alpha[f][R]$. This shows that C is unbounded below α, whence $\alpha \in C$.

Let $\theta(\alpha, f, R)$ denote the following:

$$L_\alpha[f][R] \models \text{“}ZF^- \wedge \forall \alpha \in On\ \alpha \cong \omega\text{”}.$$

We have added a real R with the following property:

$$\forall \alpha, \beta < \omega_1 \quad (\ \theta(\alpha, f, R) \wedge \theta(\beta, f, R)\ ) \ \Rightarrow\ g_\alpha(n) = g_\beta(n) \tag{8}$$

What is the complexity of this statement? Remember that the g_ξ had a very simple, absolute definition. In fact, whenever $L_\eta[f] \models$“$\xi \cong \omega$” and $L_\eta[f]$ is a model of, say Δ_1-Replacement (which makes $<_L$ definable inside $L_\eta[f]$), $g_\xi \in L_\eta[f]$ and is definable there. So if α, β are countable in $L_\eta[f]$, a model of at least Δ_1-Replacement, $L_\eta[f] \models$“$g_\alpha(n) = g_\beta(n)$” is equivalent to a statement that is Δ_1 in the parameters α, β, n and f. So (8) can be written as:

$$\begin{gathered}\forall \alpha, \beta, \gamma < \omega_1 \\ [\theta(\alpha, f, R) \wedge \theta(\beta, f, R)\ \wedge\ \theta(\gamma, f, R)\ \wedge\ \gamma > \alpha, \beta\ ] \Rightarrow \\ L_\gamma[f] \models \text{“}g_\alpha(n) = g_\beta(n)\text{”})\end{gathered}$$

Denote this statement by $\Psi(R, n, f)$. By the arguments of the preceding paragraph this statement is Π_1 in R, n and f over HC, and $\exists R \Psi(R, n, f)$ is $\Sigma_2(f, n)$ over HC. Take $r_f \in 2^\omega$ coding f; we can find an equivalent $\Sigma^1_3(r_f)$ sentence $\exists r \bar{\Psi}_n(r_f)$. As $V[G][R] \models \exists r \bar{\Psi}_n(r_f)$ (where $r_f \in V$) and G and R where obtained by a finite iteration of ω_1-preserving forcings, the aboluteness assumption yields $V \models \exists r \bar{\Psi}_n(r_f)$, and hence $V \models \exists R \Psi(R, n, f)$.

We work in V again. We have shown, that for all $n \in \omega$, there exists a witness R_n to (8). So let, for each $n \in \omega$,

$$C_n := \{\ \alpha < \kappa \mid L_\alpha[f][R_n] \prec L_{\omega_1}[f][R_n]\ \}$$

By corollary 1.4, these are *cub* subsets of ω_1. For any $n \in \omega$ and $\alpha, \in C_n$, $\theta(\alpha, f, R_n)$ holds (by elementarity). So, as (8) holds for each R_n, we have

$$\forall n \in \omega \quad \forall \alpha, \beta \in C_n \quad g_\alpha(n) = g_\beta(n).$$

As $\bigcap_{n \in \omega} C_n$ is *cub*, we can choose distinct $\alpha, \beta \in \bigcap_{n\in\omega} C_n$. Now as promised, $g_\alpha = g_\beta$, a contradiction. $\square$

3.2 Corollary. *Assume $\mathbf{\Sigma^1_3}$-absoluteness for the class of all ω_1-preserving set forcings. Then, ω_1 is inaccessible to reals, i.e. for $\kappa = \omega_1^V$ and any $r \in 2^\omega$, $L[r] \models$ “κ is inaccessible”.*

Proof. Start with an r s.t. $L[r] \models \kappa = \lambda^+$. Repeat the argument above, dragging along r as a parameter, leading to the same contradiction. $\square$

4 Coding and reshaping when ω_1 is inaccessible to reals

In this section we give proofs of:

1. [Fri] $\mathbf{\Sigma^1_3}$ absoluteness for ω_1-preserving forcing is equiconsistent with the existence of a reflecting cardinal.
2. [Fri] "$\mathbf{\Sigma^1_3}$ absoluteness for proper forcing holds and ω_1 is inaccessible to reals" is equiconsistent with a reflecting cardinal.

How to obtain a model for $\mathbf{\Sigma^1_3}$ absoluteness for all set forcings from a reflecting has already been shown. It remains to show how to obtain large cardinal strength from the requisite absoluteness assumption. The first claim is almost implicit in [FB01]. In [Sch00a] it is shown that the forcing used in [Fri] to prove the first claim is in fact stationary preserving, leading to the intermediate result for the class of stationary-preserving set forcing. The second claim builds on work in [Sch00b].

4.1 Definition. Let $\alpha < \beta$ be regular cardinals. We say *β is α-reflecting*, $\iff$ for any formula ϕ with parameters in H_α, if there is a cardinal λ s.t. $H_\lambda \models \phi$, then there is such a λ below β.

In an earlier attempt to obtain large cardinal consistency strength from $\mathbf{\Sigma^1_3}$-absoluteness for ω_1-preserving set forcing, it was proved it implies that ${\omega_2}^V$ is ${\omega_1}^V$-reflecting in L. But this is consistency-wise not stronger than ZFC.

4.2 Fact. Let α be a cardinal, β a regular cardinal $\geq \alpha$. Then $Con(ZFC + (\gamma = \beta^+ \Rightarrow L \models$"$\gamma$ is α-reflecting"$)) \iff Con(ZFC)$

Proof. For each formula $\phi(x_1, .., x_k)$ and each $\vec{p} = (p_1, .., p_k) \in [L_\alpha]^{<\omega}$, choose $\lambda(\phi, \vec{p})$ s.t. $L_{\lambda(\phi,\vec{p})} \models \phi(p_1, .., p_k)$, if such λ exists. Let κ be some regular cardinal greater then all the $\lambda(\phi, \vec{p})$. Assuming $\kappa > \beta^+$, force with $Coll(\beta, \{\kappa\})$, defined in 1.24, p. 11. By fact 1.25, $Coll(\beta, \{\kappa\})$ preserves α and κ^+ remains a cardinal. By construction, κ^+ is α-reflecting in L, and this still holds in the extension, but there, $\kappa^+ = \beta^+$. □

4.1 Preserving ω_1

4.3 Theorem. *If $\mathbf{\Sigma^1_3}$-absoluteness holds for ω_1-preserving set forcing, ${\omega_1}^V$ is reflecting in L.*

Proof. Let ϕ be any statement with parameters $\vec{p}$ in $L_{\omega_1} = H_{\omega_1} \cap L = (H_{\omega_1 V})^L$, and fix λ s.t. $L_\lambda \vDash \phi$ ($(H_\lambda)^L = L_\lambda$, as λ is an L-cardinal). We shall show, using the absoluteness assumption, there is such a λ below ω_2. Let me first try to give a sketch of the problems encountered.

We will first collapse λ and then we will force to get a real witnessing, in a sense, that there is a λ below ω_2 with the desired properties. If we manage to ensure the witnessing via a $\mathbf{\Pi^1_2}$ property of the real, by our absoluteness assumption, such a witness will exist in the ground model. When we "decode" λ from the real, we obtain $\lambda < \omega_1$ with the desired properties, and this makes crucial use of the previously observed fact that ω_1 is inaccessible to reals.

The first difficulty arises in the "coding" process: of course, after collapsing λ to ω_1, we could code it by a set $A \subseteq \omega_1$, and code this by a real using almost disjoint forcing. But we want to be able to decode in the ground model. So we shall use an almost disjoint family recursively constructible from initial segments of A. Such a thing exists if we reshape A, using Jensen's classical forcing from [BJW82], but to show the reshaping-forcing preserves ω_1, we will need A to code much more information, in fact, it will need to code all of H_{ω_2} of the model where λ has been collapsed. This problem can be solved as we may assume $0^\sharp$ does not exist.

The second problem is to ensure our real will do its witnessing job via a $\mathbf{\Pi^1_2}$-statement, which seems difficult, as it witnesses a property of uncountable sets. A variant of the reshaping-forcing will be able to deal with this. Now let's get on with the proof.

Collapsing and coding

First of all, we may assume $0^\sharp$ does not exist, as otherwise, any uncountable V-cardinal would be rather large in L, by all means a true reflecting cardinal. For if $0^\sharp$ exists, for any uncountable cardinals $\alpha < \beta$, we have $L_\alpha \prec L_\beta$. Using reflection we can see $L_{\omega_1} \prec L$, which clearly implies ω_1 is reflecting by Fact 1.39.

Take κ a strong limit singular above λ, with uncountable cofinality. As $0^\sharp$ doesn't exist, we are allowed to use the Covering Lemma (see Fact 1.36, p. 18), whence $2^\kappa = \kappa^+$, and $\kappa^+ = (\kappa^+)^L$, by Fact 1.37.

Now we force with $Coll(\omega_1, \{\kappa\})$ (see definition 1.24), temporarily denoted by Col. We have $\kappa^\omega = \kappa$, so by fact 1.25, Col has the $\kappa^+ - cc$, whence $\kappa^+ = \omega_2^{V^{Col}} = (\kappa^+)^L$ (so κ is just an ordinal between ω_1 and ω_2 in the extension). Moreover, as Col is σ-closed, $\mathcal{P}(\omega)^{V^{Col}} = \mathcal{P}(\omega)^V$ and has size less than κ, so Col forces the continuum hypothesis. Until further notice, we work in V^{Col}.

Any $x \in H_{\omega_2}$ can be coded by a well founded relation E_x on ω_1, so we can find a set $A \subseteq \omega_2$ s.t. $H_{\omega_2} \subseteq L[A]$, as there are only ω_2 subsets of ω_1: Let $S \subseteq \omega_1$. Then S has a name $\dot{S} \in V$, of the form $\bigcup_{\chi<\omega_1} \{\check{\chi}\} \times D_\chi$, where each $D_\chi \subseteq Col$. But in V, there are at most $\mathcal{P}(Col)^{\omega_1} \cong 2^{\kappa\cdot\omega_1} \cong \kappa^+$ such names. So in V^{Col}, $2^{\omega_1} = \kappa^+ = \omega_2$. Actually, $H_{\omega_2} \subseteq L_{\omega_2}[A]$. Let $G \subseteq \omega_1$ code a surjection from ω_1 onto κ. As $\kappa^+ = (\kappa^+)^L$, $L[G] \vDash {\omega_1}^+ = \kappa^+$. So we can choose an almost disjoint family $\mathcal{A}$ on ω_1 of size κ^+, $\mathcal{A} \in L[G]$. Now we force with the almost disjoint coding $P_{\mathcal{A},A}$ (see section 1.2, p. 6). $P_{\mathcal{A},A}$ is σ-closed. From now on we work in $W := V^{Col * P_{\mathcal{A},A}}$.

Take $A' \subseteq \omega_1$ such that it codes both G and the generic for $P_{\mathcal{A},A}$. Then we have $A \in L[A']$, and so $(H_{\omega_2})^{V^{Col}} \subseteq L_{\omega_2}[A']$. Note that even $(H_{\omega_2})^W \subseteq L_{\omega_2}[A'] = (H_{\omega_2})^{L[A']}$: For $x \in H_{\omega_2}$, let $E_x \subseteq \omega_1$ code $\in\restriction TC(\{x\})$. This E_x has a "nice name" in V^{Col}, of the form $\dot{E}_x = \bigcup_{\chi\in\omega_1} \{\chi\} \times A_\chi$, where each A_χ is an antichain. $P_{\mathcal{A},A}$ has the ω_2-cc, as the set of first components of conditions has size ${\omega_1}^{<\omega_1} = \omega_1$. Thus $\dot{E}_x \in (H_{\omega_2})^{V^{Col}} \subseteq L[A']$, and so $E_x = \dot{E}_x[A'] \in L[A']$. Also note that by σ-closedness of $P_{\mathcal{A},A}$, CH still holds in W.

Reshaping

Now that we have found a sufficiently smart $A' \subseteq \omega_1$, we can set to the task of coding it into a real R, again using almost disjoint coding. But when we decode in the ground model V, we can only rely on decoding $A' \cap (\omega_1)^L$, having no larger a.d. family easily accessible from both models. It would help if we had $L[A' \cap \xi] \vDash \xi \cong \omega$, for all $\xi < \omega_1$. This property of A' is called "reshaped". We use the following forcing:

$$\begin{array}{rlll} P := & \{s \overset{bnd}{\subset} \omega_1| & \text{(i)} & \forall\, \xi \in lim\, \forall n, \text{ if } \xi + 2(n+1) \leq sup(s), \text{ then} \\ & & & \xi + 2(n+1) \in s \iff \xi + n \in A' \\ & & \text{(ii)} & \forall\, \xi \leq sup(s) \quad L[s \cap \xi] \vDash \text{“}\xi \cong \omega\text{”}\}, \end{array}$$

ordered by end extension. Let's show this forcing achieves what is promised. For each $\alpha < \omega_1$, the set $D_\alpha := \{s \in P | \alpha \leq sup(s)\}$ is dense in P: Let p_0 be a condition with supremum δ, and let $\alpha < \omega_1$. Let $E \subseteq \{\delta+2n+1 | n \in \omega\}$ code the epsilon relation on α. Now let $p := p_0 \cup E \cup \{\xi+2(n+1) | \xi+n \in A\} \cap \alpha+1$. This is a condition, hits D_α and extends p_0. By this density argument, we know that the generic for P will be unbounded in ω_1. Observe that the definition has been set up so that the following, later to be used fact holds:

4.4 Corollary. *For all $p \in P$ and for any $\alpha < \omega_1$, there is $q \leq p$ with $\alpha \leq sup(q)$ and $(q - p) \cap Lim = \emptyset$*

4.5 Lemma. *The Reshaping forcing P is ω_1-distributive.*

Proof of lemma. Now we shall use the fact that A' codes all of H_{ω_2}. Let $(D_n)_{n\in\omega}$ be a sequence of dense subsets of P, and let p_0 be a condition. We will find $p_\omega \in \bigcap_{n\in\omega} D_n$ extending p_0. First of all, observe $\{P, (D_n)_{n\in\omega}, p_0\} \subseteq H_{\omega_2} \subseteq L_{\omega_2}[A']$. Let $\mathcal{N}$ denote $\langle L_{\omega_2}[A'], P, (D_n)_{n\in\omega}, p_0\rangle$ for now. Inductively build a countable chain of elementary submodels:

$$\begin{array}{rcl} M_0 & := & <_{L[A']}\text{-least countable s.t. } M_0 \prec \mathcal{N} \\ M_{n+1} & := & <_{L[A']}\text{-least countable s.t. } M_{n+1} \prec \mathcal{N}\text{, and } M_n \in M_{n+1}. \\ M_\omega & := & \bigcup_{n\in\omega} M_n \text{ (whence } M_\omega \prec \mathcal{N}\text{)} \end{array} \tag{9}$$

Now, for $i \in \omega+1$, let π_i be the inverse of the map collapsing M_i to a transitive set. The collapse of M_i is of the form $\langle L_{\delta_i}[A' \cap \beta_i], P^i, (D^i_n)_{n\in\omega}, p^i_0\rangle$, where $\beta_i, \delta_i < \omega_1$, $\{P^i, (D^i_n)_{n\in\omega}, p^i_0\} \subseteq L_{\delta_i}[A' \cap \beta_i]$ and each of these predicates is mapped by π to its corresponding set in $\mathcal{N}$, e.g. $\pi_i(P^i) = P$. Using elementarity, we get that $\pi_i(\beta_i) = \omega_1$, and β_i is the least ordinal moved by π_i. So as $p^i_0 \subseteq \beta_i$, $p^i_0 = p_0$ (using the fact that π is the identity on β_i and elementarity). See the figure for an overview of the situation.

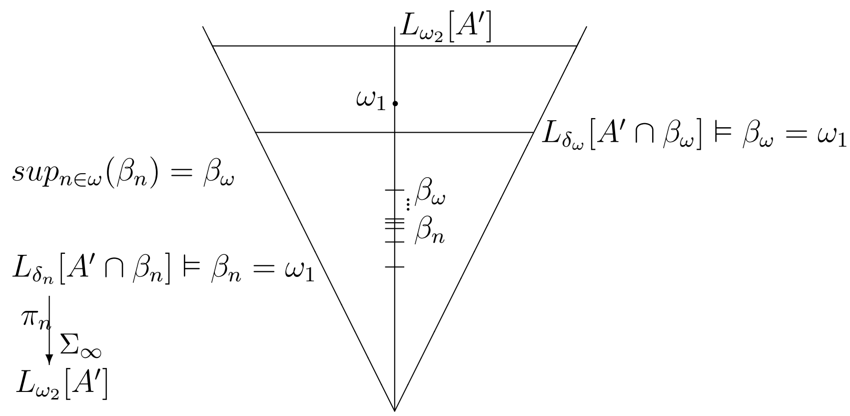

Figure 1: Traces of a countable chain of elementary submodels

As $ran\pi_n \subseteq ran\pi_\omega$,

$$\langle L_{\delta_n}[A' \cap \beta_n], P^n, (D^n_m)_{m\in\omega}, p^n_0\rangle \xrightarrow[\Sigma_\omega]{{\pi_\omega}^{-1} \circ \pi_n} \langle L_{\delta_\omega}[A' \cap \beta_\omega], P^\omega, (D^\omega_m)_{m\in\omega}, p^\omega_0\rangle \tag{10}$$

for each $n \in \omega$. Call the model on the right hand side of (10) $\mathcal{N}'$, and let $M'_n := ran(\pi_n \circ {\pi_\omega}^{-1})$. Then we can write (10) as $M'_n \prec \mathcal{N}'$. Now we can see, using elementarity and absoluteness of $<_{L[A']}$, the output of the definition (9), with $\mathcal{N}$ replaced by $\mathcal{N}'$, is just the chain $(M'_n)_{n\in\omega}$. Thus, the sequences $(\delta_n)_{n\in\omega}$, $(\beta_n)_{n\in\omega}$ are elements of $L_{\omega_2}[A' \cap \beta_\omega]$.

Now choose $p_{n+1} \in M_{n+1} \cap D_n$ inductively for all $n \in \omega$, so that $p_{n+1} \leq p_n$ and $sup(p_{n+1}) > \beta_n = \omega_1 \cap M_n$ (this is possible as β_n is countable in M_{n+1}). Let

$$p_\omega := \bigcup_{n\in\omega} p_n.$$

These p_i are not moved by π_j, for $j \geq i$ in $\omega+1$. The only thing left to check is that p_ω is a condition. The part about coding A' is clear. Let $\xi < sup(p_\omega)$. Then $L[p_\omega \cap \xi] = L[p_n \cap \xi]$, for some n, so in this case the desired property holds as p_n is a condition.

Now let $\xi = sup(p_\omega)$. Then, by construction of p_ω, $\xi = \beta_\omega$. So $L[p_\omega] \supseteq L[A' \cap \beta_\omega]$, and the latter model knows that β_ω is the limit of a countable sequence of countable ordinals, namely $(\beta_n)_{n\in\omega}$, hence β_ω is countable in $L[p_\omega]$. □

From now on, we shall work in W^P, containing A'', the P-generic, which is reshaped. Of course A' and thus $(H_{\omega_2})^W$ are constructible from A''. By exactly the same argument as for the almost disjoint coding, again $(H_{\omega_2})^{W^P} \subseteq L_{\omega_2}[A'']$, since P consists of hereditarily countable sets while CH holds.

Killing universes

We are now almost ready for coding A'' into a real. But first we must tackle the second problem mentioned at the beginning. Among other things, we have achieved that

$$L_{\omega_2}[A''] \vDash \text{“} \exists\lambda \in Card^L \text{ s.t. } L_\lambda \vDash \phi(\vec{p}\,)\text{”} \tag{11}$$

Let “ $\exists\lambda \in Card^L$ s.t. $L_\lambda \vDash \phi(\vec{p}\,)$” be denoted by $\psi(\vec{p}\,)$. We can w.l.o.g. assume that any transitive ZF^- model containing A'' will believe $\psi(\vec{p}\,)$ (by changing A'' such that part of it codes the epsilon relation on λ). Then we have:

$$\text{for all transitive } M,\ \big(M \vDash ZF^- \wedge\ \{\omega_1, A''\} \subseteq M\big) \Rightarrow M \vDash \psi(\vec{p}\,)\big) \tag{12}$$

To motivate our next step, let me jump ahead: in order to apply our absoluteness assumption, we shall use a statement very much like the one above, but only mentioning reals. To eventually formulate such a statement, it would be nice if we didn't have to quantify over all M, but only over members of HC. In fact we can, by an ω_1 preserving forcing, make a stronger statement than (12) true, where quantification is over HC. We will use the following forcing notion:

$$
\begin{array}{lll}
P' := & \{p \overset{bnd}{\subset} \omega_1 | & \text{(i)} \quad \forall\, \xi \in lim\, \forall n, \text{ if } \xi + 2(n+1) \leq sup(s), \text{ then} \\
& & \qquad \xi + 2(n+1) \in s \iff \xi + n \in A' \\
& & (ii) \quad \forall M \in HC \\
& & \qquad \text{if } M \text{ is a transitive model of `` } ZF^- + \exists\, \omega_1 \text{ '',} \\
& & \qquad p \cap \omega_1^M \in M \text{ and } sup(p) \geq \omega_1^M, \text{ then} \\
& & \qquad M \vDash \psi(\,\vec{p}\,) \quad \},
\end{array}
$$

ordered by end-extension.

Let's show that the generic for P' is unbounded in ω_1, i.e., that for each $\alpha \in \omega_1$, the set $D_\alpha := \{s \in P' | \alpha \leq sup(s)\}$ is dense in P'. But this works just like for the Reshaping: Given p_0 and α, we just append a "short" set E coding α to obtain p_1. Any countable transitive ZF^--model M with $p_1 \in M$ that has an $\omega_1 > sup(p_0)$ must contain E, and thus everything up to and including α is countable in M. We can see that we're in a way reshaping A'' again to make ordinals that are ω_1 in models containing initial segments of A'' and believing the wrong things (that is, $\neg\psi(\,\vec{p}\,)$) increasingly sparse. Again, we would like to call the following fact to your attention, as we shall re-use it in the next section:

4.6 Corollary. *For all $p \in P'$, and for any $\alpha < \omega_1$, there is $q \leq p$ with $\alpha \leq sup(q)$ and $(q - p) \cap Lim = \emptyset$*

Now assume B is the generic for this forcing notion. Then the following improvement of (12) holds:

$$
\begin{array}{c}
\text{for all transitive } M \text{ in } HC, \\
(M \vDash \text{``}ZF^- + \exists\, \omega_1\text{''} \ \wedge\ B \cap {\omega_1}^M \in M) \Rightarrow M \vDash \psi(\,\vec{p}\,)
\end{array}
\tag{13}
$$

We still have to show that P' is ω_1-preserving, but the proof is very similar to that of lemma 4.5.

4.7 Lemma. *The Killing Universes-forcing P' is ω_1-distributive.*

Proof of lemma. Again, let $(D_n)_{n\in\omega}$ be a sequence of dense subsets, $p_0 \in P'$. Construct $(\delta_i)_{i\in\omega+1}$, $(\beta_i)_{i\in\omega+1}$ and $(p_i)_{n\in\omega+1}$ just like in the proof of lemma 4.5, with A' replaced by A'' and P by P'. We show p_ω is a condition. We only prove part (ii).

Let $M \cong \omega$, transitive, $M \vDash$ "$ZF^- + \exists\omega_1$". As before, if ${\omega_1}^M < sup(p_\omega) = \beta_\omega$, (ii) clearly holds as then for some n, ${\omega_1}^M < sup(p_n)$ and p_n is a condition. What if ${\omega_1}^M = sup(p_\omega) = \beta_\omega$? By (11) and by elementarity,

$$
L_{\delta_\omega}[A'' \cap \beta_\omega] \vDash \text{`` } \exists\lambda \in Card^L\ L_\lambda \vDash \phi \text{ ''}
\tag{14}
$$

Moreover, we have asked that A'' code λ in some simple absolute way, so the same is true of $A'' \cap \beta_\omega$ and λ_ω, whence $\lambda_\omega \in M$. It only remains to show that $M \vDash \lambda_\omega \in Card^L$. By (14), it remains to show that $M \cap On \leq \delta_\omega$. Suppose not. Using elementarity we see the images of $(D_n)_{n\in\omega}$, p_o and P' under π_ω are all constructible from $A'' \cap \beta_\omega$ by stage $\delta_\omega \in M$, and thus all the ingredients needed to define $(\delta_n)_{n\in\omega}$, $(\beta_n)_{n\in\omega}$ are present in M. But then, once more, $M \vDash \beta_\omega \cong \omega$, contradiction. So $M \cap On \leq \delta_\omega$. This shows (ii) holds for p_ω and completes the proof of the lemma. $\square$

Pulling back by a real

From now on we work in $W^{P*P'}$ where $B \subseteq \omega_1$ is generic for P'. B is still reshaped since A'' can be recovered via a simple mapping. Now we use the reshapedness to construct an a.d.-family $\mathcal{B}$: Start with the $<_L$-least real belonging to an a.d. family in L. Having constructed $(b_\xi)_{\xi\in\alpha} \in L[A'' \cap \alpha]$, we take

$$b_\alpha := \text{ the } <_{L[B\cap\alpha]}\text{-least set a.d. from } (b_\xi)_{\xi\in\alpha} \text{ in } L[B \cap \alpha],$$

which exists as $L[B \cap \alpha] \vDash \alpha \cong \omega$. Let $\mathcal{B} := (b_\xi)_{\xi\in\omega_1}$.

Now we force with $P_{\mathcal{B},B}$. Call the generic R and let us from now on work in $W' := W^{P*P'*P_{\mathcal{B},B}}$. This real R will be the witness to the fact that a λ with the desired properties exists. But in order to prove a real exists in the ground model with the same witnessing capacity, we have to express this as a $\mathbf{\Pi^1_2}$ property of R. If $\alpha < \omega_1$ and $L_\alpha[R] \vDash$ "$ZF^- \wedge \delta = \omega_1$", we can decode $B \cap \delta$ in that model, so by (13), the following holds of R:

$$\begin{gathered} \forall \alpha < \omega_1 \\ L_\alpha[R] \vDash \text{“}ZF^- + \exists\, \omega_1\text{”} \;\; \Rightarrow L_\alpha[R] \vDash \psi(\,\vec{p}\,) \end{gathered} \tag{15}$$

The formula above is clearly $\Pi_1(R, \vec{p}\,)$ over HC, so by Fact 1.33, it is equivalent to a $\Pi^1_2(R, r_p)$-formula $\Phi(R, r_p)$, for some real r_p in the ground model V.

So finally, after forcing with a finite iteration of ω_1-preserving forcing notions, there exists a real R satisfying (15). By $\mathbf{\Sigma^1_3}$-absoluteness, there is a real R_0 with the same property in V. Now we work in V again. By corollary 3.2, $\delta := {\omega_2}^{L[R_0]} < {\omega_1}^V$. $L_\delta[R_0] \vDash$ "$ZF^- + \exists\, \omega_1$". But then (15) implies that

$$L_\delta[R_0] \vDash \psi(\,\vec{p}\,)$$

i.e. $\exists \lambda \in Card^L \cap \omega_2$ s.t. $L_\lambda \vDash \phi(\,\vec{p}\,)$. $\square$

4.2 Preserving stationary subsets of ω_1

All the forcings in the previous proof where *ccc* or σ-closed, except for Reshaping and Killing Universes. These were shown to be ω_1-closed. In fact, in [Sch00a] it is shown they preserve stationary subsets of ω_1.

4.8 Lemma. *Killing Universes and Reshaping are stationary-preserving.*

Proof. We shall be working in a context where there is $A \subseteq \omega_1$ s.t. $H_{\omega_2} \subseteq L_{\omega_2}[A]$ and CH holds. We start with the Reshaping forcing P. W.l.o.g. we can assume that for all $\alpha \in lim \cap \omega_1$, $L[A \cap \alpha] \models \omega_1 \geq \alpha$, by padding A with sets $E \subseteq [\xi, \xi+\omega)$ which collapse ξ. Let $S \subset \omega_1$, stationary, and let $q \in P$, $\dot{C}$ a P-name such that $q \Vdash$"$\dot{C}$ is *cub* in ω_1". We need to find a condition $p \leq q$ s.t. $p \Vdash$ "$\dot{C} \cap \check{S} \neq \emptyset$".

By replacing $\dot{C}$ with a "nice name" if necessary, we can assume $\dot{C} \in L_{\omega_2}[A]$. Let $\mathcal{N}$ denote $\langle L_{\omega_2}[A], \dot{C}, S, P, q\rangle$. Consider the *cub* set

$$C_0 := \{\alpha < \omega_1 \mid h^{\mathcal{N}}_{\Sigma_\omega}(\alpha) \cap \omega_1 = \alpha\}$$

Let $\alpha \in C_0 \cap S$, s.t. $\alpha > sup(q)$ and set $M := h^{\mathcal{N}}_{\Sigma_\omega}(\alpha)$. There is $\pi : L_\beta[A \cap \alpha] \xrightarrow{\cong} M \prec L_{\omega_2}[A]$, where $\pi(\alpha) = \omega_1$ and α is the least ordinal moved. We should mention that (as P is definable and by elementarity) $\pi^{-1}(P) = P^{L_\beta[A\cap\alpha]} = P \cap L_\alpha[A \cap \alpha]$ and that π is the identity on that set. Observe that $(q \Vdash \pi^{-1}(\dot{C})$ is *cub* in $\omega_1)^{L_\beta[A\cap\alpha]}$, by elementarity. Thus (by taking preimages of the corresponding set under π),

$$\begin{gathered}\text{for all } \gamma < \alpha \text{, the set}\\ D_\gamma := \{p \in \pi^{-1}(P) \mid \exists \xi \in (\gamma, \alpha) \quad p \Vdash \xi \in \dot{C}\}\\ \text{is dense in } \pi^{-1}(P).\end{gathered} \tag{16}$$

We inductively build a decreasing chain $(p_i)_{i\in\omega}$, picking conditions from $P \cap L_\alpha[A \cap \alpha]$, such that $p_\infty = \bigcup_{i\in\omega} p_i \Vdash$ "$\alpha \in \dot{C}$". We start with $p_0 = q$ (observe $q \in L_\alpha[A \cap \alpha]$).

First, we consider the simple case where α happens to be countable in $L[A \cap \alpha]$: we just choose each p_{n+1} end-extending p_n, such that $p_{n+1} \Vdash \check{\xi} \in \dot{C}$ for some $\xi \in (sup(p_n), \alpha)$ (which is possible by (16)), ensuring that $\bigcup_{n\in\omega} sup(p_n) = \alpha$. It is easy to see that p_∞ is a condition, as the required property was assumed to magically hold. Clearly $p_\infty \Vdash$"$\dot{C}$ is unbounded in α", whence $p_\infty \Vdash$"$\alpha \in \dot{C}$".

If $\alpha = {\omega_1}^{L[A\cap\alpha]}$, we have to choose our sequence of conditions more carefully. Using (16), the set

$$
\begin{aligned}
C_1 := \{\eta < \alpha \mid \quad & p_0 \in L_\eta[A \cap \eta] \\
& \text{ and} \\
& \forall p \in L_\eta[A \cap \eta] \cap P \\
& \exists q \in L_\eta[A \cap \eta] \cap P \text{ s.t. } q \leq p \text{ and} \\
& \exists \xi \in (sup(p), \alpha) \quad q \Vdash \check{\xi} \in \dot{C}\ \}
\end{aligned}
$$

is easily seen to be *cub* in α, and $C_1 \in L_\beta[A \cap \alpha]$. Moreover, by thinning out, we can assume that $C_1 \subseteq lim$ and

$$\text{for all } \gamma \in C_1, \text{ if } \eta \text{ is least in } C_1 \text{ above } \gamma, \text{ then } L_\eta[A \cap \eta] \vDash \gamma \cong \omega. \tag{17}$$

Now we pick $\{\beta_i | i \in \omega\}$, cofinal in C_1 with order type ω. Assume we have already chosen $p_n \in L_{\beta_n}[A \cap \beta_n]$. Let η be the minimum of $C_1 - (\beta_n + 1)$. By corollary 4.4 and (17), we can choose $p'_n \leq p_n$ in $L_\eta[A \cap \eta]$ such that

$$(p'_n - p_n) \cap Lim = \{\beta_n\}. \tag{18}$$

Now extend p'_n to $p_{n+1} \in L_\eta[A \cap \eta]$ so that for some $\xi > sup(p_n)$, $p'_n \Vdash \check{\xi} \in \dot{C}$. Observe that

$$(p_{n+1} - p'_n) \cap C_1 = \emptyset. \tag{19}$$

Clearly, $p_{n+1} \in L_{\beta_{n+1}}[A \cap \beta_{n+1}]$, as $\eta \leq \beta_{n+1}$. Having built this sequence, if we can show p_∞ is a condition, we are done: just like in the simpler construction, $p_\infty \Vdash$ "$\alpha \in \dot{C}$". By construction, $sup(p_\infty) = \alpha$. To show $p_\infty \in P$, we concentrate on the non-trivial case and show $L[p_\infty] = L[A \cap \alpha, p_\infty] \vDash \alpha \cong \omega$. Observe that by (19) and (18), $C_1 \cap p_\infty \cap Lim = \{\beta_n \mid \ n \in \omega\}$. But $C_1 \in L_\beta[A \cap \alpha]$, and so $\{\beta_n \mid \ n \in \omega\} \in L_\beta[p_\infty]$.

The proof can be modified to show Killing Universes preseres stationary subsets of ω_1, along the lines of 4.7. Having done the same construction as above for P', we need to make sure p_∞ is a condition. Again, we only need to check the case of a countable transitive model $N \vDash$ "$\mathrm{ZF} + \exists\omega_1$" with ${\omega_1}^N = sup(p_\infty)$. By the same arguments as in the proof of 4.7, there is $\lambda \in N$ s.t. $N \vDash \Phi$ and $L_\beta[A \cap \alpha] \vDash$ "$\lambda \in$Card", and we are done if $N \cap On < \beta$. But this is clear as α is collapsed in $L_\beta[p_\infty]$. □

4.9 Corollary. *If $\mathbf{\Sigma^1_3}$-absoluteness holds for stationary-preserving set forcing and ω_1 is inaccessible to reals, ${\omega_1}^V$ is reflecting in L.*

Proof. We have just shown that all the forcings used in the proof of theorem 4.3 were stationary-preserving. At the end of the proof instead of appealing to the strength provided by absoluteness for ω_1-preserving forcings (corollary 3.2), we use the additional assumption that ω_1 is inaccessible to reals. □

4.3 Proper forcing

Remarkable cardinals were devised in [Sch00b]. They provide an exact measure for the consistency strength of "reshaping is proper". We will use the fact that, assuming ω_1 is not remarkable in L, Reshaping (and, as it turns out, also Killing Universes) are proper. For the proof, it is convenient to change the definition of Killing Universes a bit:

4.10 Definition. We define the *Killing Universes for A* to be the following forcing notion:

$$\begin{array}{lll} K.U.^A := \{p \overset{bnd}{\subset} \omega_1| & \text{for all countable transitive } M\text{, if} & \\ & M \vDash \text{“} ZF^- \wedge \exists\, \omega_1 \text{”} & \\ & \wedge\, {\omega_1}^M \leq sup(p), & \\ & \wedge\, \{A \cap {\omega_1}^M, p \cap {\omega_1}^M, \vec{p}\} \subseteq M & \\ & \wedge\, (L[A \cap \omega_1])^M \vDash \neg\exists\omega_2, & \\ & \text{then } M \vDash \text{“}\exists\lambda \in Card^L\ L_\lambda \vDash \psi(\vec{p})\text{”} & \}, \end{array}$$

ordered by end-extension.

4.11 Lemma. *For $A \subseteq \omega_1$ s.t. $L_{\omega_2}[A] = H_{\omega_2}$, if there is $\theta \in Reg \cap \omega_2$ such that (κ is not θ-remarkable)^L, where $\kappa = {\omega_1}^V$, then Reshaping and Killing Universes for A are proper.*

Proof. We shall prove the lemma in two steps. In the first step (following [Sch00b]), we show that there is a *cub* subset C of $[L_{\omega_2}[A]]^\omega$ such that for all $X \in C$, if $X = ran(\pi)$ for $\pi : L_\beta[A \cap \alpha] \to_{\Sigma_\omega} L_{\omega_2}[A]$, then β is not a regular cardinal in L.

Look at $C^\star$ equal to the *cub* set of countable $X \prec \mathcal{N} := \langle L_{\omega_2}[A], A\rangle$ such that $\theta \in X$. Assume that $X \in C^\star$, π, α and $\beta \in Reg^L$ witness failure of what is claimed above. Set $\bar\theta := \pi^{-1}(\theta)$. Clearly, $\alpha = crit(\pi)$ and $\pi(\alpha) = \kappa$. By elementarity of π and relativizing formulas to L, we can see $\bar\pi := \pi \restriction L_{\bar\theta}$ is an elementary embedding from $L_{\bar\theta}$ into L_θ, and $\bar\theta$ is regular in L (as this holds in L_β). Now let G be generic for $Coll(\omega, \kappa)$. Then (as this forcing is constructible, and in fact absolutely definable from κ), G is also $Coll(\omega, \kappa)$-generic over L. As $L[G] \vDash \bar\theta \cong \omega$, by fact 1.6, the following holds in $L[G]$:

$$\exists\bar\pi, \bar\theta \in Reg^L \quad \bar\pi : L_{\bar\theta} \to_{\Sigma_\omega} L_\theta \text{ s.t. } \bar\pi(crit(\bar\pi)) = \kappa > \bar\theta \tag{20}$$

Let this statement be denoted by $\Phi(\kappa, \theta)$. By homogeneity of the gentle collapse of κ, we have $(\Vdash_{Coll(\omega,\kappa)} \Phi(\kappa, \theta))^L$. Take a countable elementary submodel of L_{ω_2} in L: let $\gamma < {\omega_1}^L$ and $\sigma : L_\gamma \to_{\Sigma_\omega} L_{\omega_2}$, $\sigma \in L$. Let $\tilde\kappa, \tilde\theta$ denote $\sigma^{-1}(\kappa)$, $\sigma^{-1}(\theta)$, respectively. We can choose $\tilde G$ in L, $Coll(\omega, \tilde\kappa)$-generic over L_γ. By elementarity, $L_\gamma[\tilde{G}] \models \Phi(\tilde{\kappa}, \tilde{\theta})$. So there exists $\bar{\pi} \in L$ and $\bar{\theta}$ such that $\bar{\pi}$ is as in (20) and $\bar{\theta} \in Reg^{L_\gamma}$. We have found elementary embeddings $\bar{\pi}, \sigma \in L$,

$$\begin{array}{ccccc} L_{\bar{\theta}} & \xrightarrow{\bar{\pi}} & L_{\tilde{\theta}} & \xrightarrow{\bar{\sigma}} & L_\theta \\ \Big\downarrow{\scriptstyle \bar{\pi}} & & & & \\ L_{\tilde{\theta}} & & & & \end{array}$$

where $\sigma(\tilde{\kappa}) > \bar{\theta} \in Reg^{L_{\tilde{\theta}}}$. So $M := L_{\tilde{\theta}}$, σ, $N := L_{\bar{\theta}}$ and $\pi' := \sigma \circ \bar{\pi}$, witness that κ is θ-remarkable in L (definition (1.42) on page 20). This completes the first part of the proof

The second part is a version of the argument for properness of reshaping in [Sch00b], attributed there to [SS92]. It is similar to the arguments that show that reshaping and $K.U.$ are ω_1-distributive or stationary-preserving. The construction is the same for both forcing notions, so let P denote either one of them. It suffices to show that there exists C, a *cub* subset of $[L_{\omega_2}[A]]^\omega$ such that $\forall N \in C \quad \forall p \in N \cap P \quad \exists q \leq p \quad \forall \dot{\alpha} \in N$, if $p \Vdash \dot{\alpha} \in On$, then $q \Vdash \dot{\alpha} \in N$. The countable elementary submodels of $\langle [L_{\omega_2}[A]]^\omega, P\rangle$ form a *cub* set, so by taking the intersection, we can assume $C^\star$ consists of such models. Let $N \in C^\star$, $p_0 \in N$ be arbitrary. We show there is an N-generic condition below p_0.

Let $\pi : L_\beta[A \cap \alpha] \xrightarrow{\cong} N \prec L_{\omega_2}[A]]^\omega$, where $P, p_0 \in N$. We know $\pi(\alpha) = \omega_1$, $crit(\pi) = \alpha$ and $\beta \notin Reg^{L[A\cap\alpha]}$, as $N \in C^\star$. As there are no cardinals above α in $L_\beta[A \cap \alpha]$, $L[A \cap \alpha] \models \beta \cong \alpha$. So let $(\dot{\alpha}_\xi)_{\xi\in\alpha} \in L[A \cap \alpha]$ be an enumeration of all $\pi^{-1}(P)$-names $\dot{\alpha}$ in $L_\beta[A \cap \alpha]$ such that $L_\beta[A \cap \alpha] \models$"$\Vdash_{\pi^{-1}(P)} \dot{\alpha} \in On$". Some easy consequences of elementarity: $(\pi(\dot{\alpha}_\xi))_{\xi<\alpha}$ enumerates all $\dot{\alpha} \in N$ such that $\Vdash_P \dot{\alpha} \in On$. Observe that $\pi^{-1}(P) = L_\alpha[A \cap \alpha] \cap P$. Also, $L_\beta[A \cap \alpha] \models$"$q \Vdash_{\pi^{-1}(P)} \dot{\alpha} = \check{\lambda}$" exactly if $q \Vdash_P \pi(\dot{\alpha}) = \pi(\check{\lambda})$. Clearly,

$$C_\xi := \{\eta < \alpha \quad | \quad \begin{array}{ll} (i) & \forall p \in L_\eta[A \cap \eta] \cap P, p \leq p_0 \\ & \exists q \in L_\eta[A \cap \eta] \cap P, q \leq p \\ & L_\beta[A \cap \alpha] \models \text{“}\exists \lambda \quad q \Vdash_{\pi^{-1}(P)} \dot{\alpha}_\xi = \check{\lambda}\text{”} \\ (ii) & p_0 \in L_\eta[A \cap \eta] \quad \}. \end{array} \tag{21}$$

is *cub* in α. Clearly, each $C_\xi \in L_\beta[A \cap \alpha]$. Thus, $C_0 := \triangle_{\xi\in\alpha} C_\xi$ is *cub* and $C_0 \in L[A \cap \alpha]$. By thinning out C_0 if necessary, we can assume

$$\begin{array}{c} \text{for all } \gamma \in C_0, \text{ if } \eta \text{ is least in } C_0 \text{ above } \lambda, \text{ then} \\ L_\eta[A \cap \eta] \models \gamma \cong \omega \end{array} \tag{22}$$

Now we build a countable chain of conditions, starting with p_0 and working in $L_{\omega_2}[A]$, but picking conditions from $L_\beta[A \cap \alpha]$. For this purpose, choose a bijection $f : \omega \to \alpha$, and choose $\{\beta_j | j \in \omega\}$, a cofinal subset of C_0 of order type ω.

Assume we have chosen $p_n \in L_{\beta_n}[A \cap \beta_n]$. Let η be least in $C_0 - (\beta_n + 1)$. By Corollary 4.4 (or 4.6 for K.U.) together with (22), we can pick $p'_n \leq p_n$ in $L_\eta[A \cap \eta]$ such that

$$(p'_n - p_n) \cap Lim = \{\beta_n\}. \tag{23}$$

Now we extend p'_n to p_{n+1}, working in $L_\eta[A \cap \eta]$, such that for all k satisfying both $k \leq n$ and $f(k) < \eta$, $p_{n+1} \Vdash_P \pi(\dot{\alpha}_{f(k)}) = \check{\lambda}$ for some $\lambda \in N$. This is possible as $\eta \in \bigcap_{\xi<\eta} C_\xi$ and by (21). Of couse $p_{n+1} \in L_{\beta_{n+1}}[A \cap \beta_{n+1}]$ (as $\eta \leq \beta_{n+1}$). Observe that

$$(p_{n+1} - p'_n) \cap C_0 = \emptyset. \tag{24}$$

Now we finish the construction by setting $p_\infty := \bigcup_{i\in\omega} p_i$. We claim p_∞ is an N-generic condition of P.

If p_∞ is a condition at all, it is N-generic: for any $\xi < \alpha$ and m such that both $\xi \in f''m$ and $\xi \leq \beta_m$, $p_m \Vdash_{\pi^{-1}(P)} \dot{\alpha}_\xi = \check{\lambda}$ for some $\lambda \in L_\beta[A]$. Hence $p_m \Vdash \pi(\dot{\alpha}_\xi) = \check{\lambda}$ for some $\lambda \in N$, and $(\pi(\dot{\alpha}_\xi))_{\xi<\alpha}$ enumerates all ordinal-names that are elements of N.

To check that p_∞ is a condition, we must treat Killing Universes and Reshaping separately.

Case 1: $P =$ Reshaping.

As usual, we restrict our attention to the non-trivial case and show $L[A \cap \alpha] \vDash \alpha \cong \omega$. By (24) and (23), for each $n \in \omega$,

$$(p_{n+1} - p_n) \cap C_0 \cap Lim = \{\beta_n\},$$

and thus

$$p_\infty \cap C_0 = \{\beta_n | n \in \omega\}.$$

As $L[A \cap \alpha] \vDash \alpha \cong \beta$, this model contains the enumeration of the *cub* sets C_ξ of (21), whence $C_0 \in L[A \cap \alpha]$. So $L[A \cap \alpha, p_\infty] \vDash \alpha \cong \omega$.

Case 2: $P =$ Killing Universes.

We need to show: For all countable transitive M which are models of $\vDash$"$\exists\omega_1 \wedge L[A \cap \omega_1] \vDash \neg\exists\omega_2$", if $\omega_1^M \leq sup(p_\infty)$ and $\{A \cap \omega_1^M, p_\infty \cap \omega_1^M, \vec{p}\} \subseteq M$, then $M \vDash$"$\exists\lambda \in Card^L\ L_\lambda \vDash \psi(\vec{p})$". Again, we only consider the case $\omega_1^M = sup(p_\infty) = \alpha$. As there exists an elementary embedding from $L_\beta[A \cap \alpha]$ into $L_{\omega_2}[A]$, the same arguments as in the proof of lemma 4.7 show there exists $\lambda \in M$ with $L_\lambda \vDash \psi(\vec{p})$, and also $\lambda \in Card^{L_\beta[A\cap\alpha]}$. Again we need to show $L_{M\cap On} \vDash \lambda \in Card$. It suffices to show $M \cap On \leq \beta$.

Assume otherwise. As $L_{M\cap On}[A \cap \alpha] \vDash \neg\exists\omega_2$, $M \cap On$ must be large enough so that $L_{M\cap On}[A \cap \alpha] \vDash \beta \cong \alpha$; but then, by the same argument as in case 1, $C_0 \in M$ and $M \vDash \alpha \cong \omega$, a contradiction. $\square$

4.12 Corollary. *[Fri] If $\mathbf{\Sigma^1_3}$-absoluteness holds for stationary-preserving set forcing and ω_1 is inaccessible to reals, ${\omega_1}^V$ is reflecting in L.*

Proof. If ω_1 is remarkable in L, we are done by fact 1.43. By fact 1.45, we can assume $0^\sharp$ does not exist. The rest is like the proof of theorem 4.3, but when you pick a strong limit singular κ to collapse, at the beginning, make sure there is a $\theta < \kappa$ witnessing that ω_1 is not remarkable in L below κ. Then all the forcing notions used in the proof are proper. In the last step, we need to use the additional assumption that ω_1 is inaccessible to reals. $\square$

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5 Forcing with the countable chain condition

5.1 Lightface Σ^1_2-indescribable cardinals

Consider a language with variables of type k for each $k \in \omega$. We say $\Phi(X_0, \dots X_k)$ is a Σ^m_n-formula if Φ starts with a block of quantifiers over variables of type m or less, with n changes of quantifier, the first quantifier being $\exists$, and only quantification over variables of type strictly less than m occurs after that. Π^m_n means negation of Σ^m_n. We define satisfaction for such higher order formulas by induction on the order m: for a Σ^m_k formula Φ,

$$\langle M, X_0, \dots, X_k \rangle \models \exists Y_0 \dots QY_r \Phi(Y_0, \dots, Y_r, X_0, \dots, X_r)$$

(where Q denotes $\exists$ or $\forall$) exactly if

$$\begin{gathered}\exists Y_0 \in \mathcal{P}^m(M) \dots QY_r \in \mathcal{P}^m(M) \\ \text{s.t. } \langle M, X_0, \dots, X_k, Y_0, \dots, Y_r \rangle \models \Phi(Y_0, \dots, Y_r, X_0, \dots, X_r)\end{gathered}$$

($\mathcal{P}^m(M)$ denotes the $m-th$ application of the power set operation); satisfaction for Σ^0_n-formulas is just normal first-order satisfaction (over a structure with additional higher order predicates) extended to atomic formulas $X \in Y$, $X = Y$ over the higher-order predicates so that they hold just if their obvious interpretations hold.

We shall use this definition for the case $m = 1$; variables of type 1 shall be distinguished from those of type 1 by using upper-case for the former, lower-case for the latter. For the structures we consider, the usual translation between sets and sets of ordinals goes through, so we shall assume that second order variables range over sets of ordinals only.

We shall also make essential use of Σ_1-definable Skolem functions for L (see [Dev84, II, 6.]).

5.1 Definition. We say κ has the Σ^1_2 reflection property if whenever $V_\kappa \models \exists X \forall Y \Phi(X, Y, p)$ for $p \in V_\kappa$ (and first-order Φ), then there is $\xi < \kappa$ such that $V_\xi \models \exists X \forall Y \Phi(X, Y, p)$. We say κ is (lightface) Σ^1_2-indescribable if in addition κ is inaccessible.

5.2 Fact.
1. If κ has said reflection property, it is a limit cardinal and is not equal to 2^λ for any $\lambda < \kappa$.
2. If there is a Mahlo cardinal, it is consistent that any *cub* class of ordinals contains a Σ^1_2-indescribable. The least Mahlo is not Σ^1_2-indescribable.

3. Reflecting implies Σ^1_2-indescribable which in turn implies the existence of many inaccessibles.

4. If P is a p.o. of size less than κ, then forcing with P preserves Σ^1_2-indescribability of κ.

Proof. 1. Express "there is no cofinal function from λ into the ordinals" and "there is a bijection from $\mathcal{P}(\lambda)$ onto the ordinals" as second order statements with parameter λ to get sentences that can't be reflected.

2. Consider the function that assigns to every set M the least ξ such that all formulas of a given class with parameters from M that hold in any V_α already hold in some V_α with $\alpha < \xi$. The closure points under this function form a *cub* class in the ordinals (these closure points have very strong reflection properties). Carry out this argument inside the initial segment of the universe below a Mahlo to get a stationary set of inaccessibles having the reflection property relativized to that initial segment. To see that the least Mahlo cannot have the reflection property, observe that κ being Mahlo is expressible by a Π^1_1 statement over V_κ.

3. For the first, observe $V_\alpha \models \exists X \forall Y \Phi(X, Y, p) \iff H_{\alpha^+} \models \exists x \forall y H_\alpha \models \Phi'(x, y, p)$, for some first-order formula Φ'. Secondly, being inaccessible is expressible as a Π^1_1 statement (f.e. the power set of any set exists and can be mapped injectively into some ordinal and there is no function with a set as domain but unbounded range).

4. Observe that if κ is inaccessible and $|P| < \kappa$, the elements of the $(H_\kappa)^{V^P}$ and $(H_{\kappa^+})^{V^P}$ are precisely the interpretations of the P-names in H_κ and H_{κ^+}, respectively. Thus $\Vdash_P$"$V_\kappa \models \exists X \forall Y \Phi(X, Y, \dot{p})$" is equivalent to a statement of the form $\exists \dot{X} \in H_{\kappa^+} \quad \forall \dot{Y} \in H_{\kappa^+} \quad V_\kappa \models$"$\Vdash_P \Phi'(\dot{X}, \dot{Y}, \dot{p})$", where Φ' is a first order expression in the parameter $P \in V_\kappa$, so the whole statement is Σ^1_2 over V_κ. So it holds with κ replaced by some inaccessible $\xi < \kappa$, and therefore $\Vdash_P$"$V_{\check{\xi}} \models \exists X \forall Y \Phi(X, Y, \dot{p})$"

$\square$

5.3 Fact. κ is lightface Σ^1_2-indescribable $\iff$ κ is inaccessible and $H_\kappa \prec_{\Sigma_2} H_{\kappa^+}$.

Proof. First assume indescribability. Let $H_{\kappa^+} \models \exists x \forall y \phi(x, y, p)$, where $p \in H_\kappa$. Pick a witness x_0 in H_{κ^+}. For any transitive M of size κ containing x_0 and p, we have $M \models \forall y \phi(x_0, y, p)$. So $H_{\kappa^+} \models \exists x \forall y \phi(x, y, p)$ is equivalent to a Σ^1_2 assertion over H_κ, and thus is reflected by some H_ξ, for inaccessible

$\xi < \kappa$. Thus $H_{\xi^+} \vDash \exists x \forall y \phi(x, y, p)$, and as Σ_2 formulas are upwards absolute for members of the H-hierarchy, $H_\kappa \vDash \exists x \forall y \phi(x, y, p)$. For the other direction, let $H_\kappa \vDash \exists X \forall Y \phi(X, Y, p)$. Then H_{κ^+} thinks "there is a cardinal θ such that $\exists x \subseteq \theta \forall y \subseteq \theta$, $\langle H_\theta, x, y\rangle \vDash (x, y, p)$". As "$y = H_\theta$" is Π_1 in y and θ, this is seen to be a Σ_2 statement in parameter p and so holds in H_κ. □

5.4 Fact. If κ is lightface Σ^1_2-indescribable, then it is both Σ_1-Mahlo and Π_1-Mahlo. If "$x \in C$" is Σ_2 over H_{κ^+} (with parameter in H_κ) for a *cub* subset C of κ, then C contains an inaccessible.

Proof. The first assertion is clearly a consequence of the second; for the latter, since $H_\kappa \prec_{\Sigma_2} H_{\kappa^+}$, the argument given in 1.41 for reflecting cardinals goes through. □

5.2 Coding using an Aronszajn-tree

We fix the following notation: for a tree T, we denote by $<_T$ (or $\leq_T$) the tree order, T_α denotes the α-th level of T and $T \restriction \alpha$ denotes the subtree of T consisting of all levels of height less than α. By $pred(t)$ we mean of course $\{t' \in T | t' <_T t\}$. The following works for any Aronszajn-tree, that is, a tree of height ω_1 with countable levels and without any cofinal branches. Let T be such a tree. Such a tree can be specialized by a *ccc* forcing, in the sense that in the extension, there exists a function F from T into the rationals which is order preserving (such F is called a specializing function). Once a tree is specialized, it is impossible to add a cofinal branch without at the same time collapsing ω_1.

5.5 Lemma. *If T is special (i.e. there exists a specializing function) and of height ω_1, T does not have a cofinal branch.*

Proof. The image of a cofinal branch under F would be a set of rationals of order type ω_1. □

A slight variation of the forcing that specializes T can be used to code a subset of ω_1 very efficiently, in a way that makes reshaping unnecessary. This approach to coding was used in [HS85] to prove that MA together with "ω_1 is inaccessible to reals" implies ω_1 is weakly compact in L (weakly compact cardinals are much stronger than reflecting). Lightface Σ^1_2-indescribability is an obvious weakening of weak compactness, which is equivalent to "boldface" Π^1_1-indescribability and closely linked to two-step Σ^1_3-absoluteness for set forcing.

5.6 Fact. Let $S = (s_\alpha)_{\alpha<\omega_1}$ be a sequence of reals. There is a *ccc* forcing P that adds a real r such that in the extension the following holds: whenever M is a transitive model of ZF^- s.t. $r \in M$, $\langle T \leq_T \rangle \in M$, we have $(s_\alpha)_{\alpha<\omega_1} \in M$.

Proof. To achieve this, we iterate the following notion of forcing: fix Q_0, Q_1, two disjoint dense sets whose union is all rational numbers. For any sequence $S = (s_\alpha)_{\alpha<\omega_1}$ consider P_T^S consisting of all conditions f s.t.

1. f is a function with domain a finite subset of $T \times \omega$
2. for each $n \in \omega$, the function $t \mapsto f(t,n)$ is a partial order preserving mapping from $(T, <_T)$ into the rationals
3. For any $\alpha < \omega_1$, t at the $\omega\cdot\alpha$-th level of T and $n \in \omega$, if $(t,n) \in dom(f)$, then $f(t,n) \in Q_0$ if and only if $n \in s_\alpha$.

5.7 Lemma. *Let $F = \bigcup G$, where G is generic. Then F is a function from T into the rationals which is order preserving and continuous at limit nodes of T; moreover, for any $\alpha < \omega_1$, and any $t \in T_{\omega\cdot\alpha}$, $\{ n \in \omega \mid F(t,n) \in Q_0 \} = s_\alpha$.*

Proof. Clearly, $D_{(t,n)} := \{ p \in P_T^S \mid (t,n) \in dom(p) \}$ is dense for any $(t,n) \in T\times\omega$: given a condition p, there is an interval of possible values for p at (t,n) (since p has finite domain), so if t is at level $\omega\cdot\alpha$ of T, we can choose a value from Q_0 or Q_1, depending on whether $n \in s_\alpha$ or not. So F is a total, order preserving function on T, and so the "moreover" clause holds by definition. F is continuous as $D_{(t,n),\epsilon} := \{ p \in P_T^S \mid \exists t' \in T \; |p(t',n) - p(t,n)| < \epsilon \}$ is dense for any $n \in \omega$, $\epsilon > 0$ and t at a limit level of T (again, by the finiteness of the domain of any condition). □

5.8 Lemma. *P_T^S is ccc.*

Proof. Assume $(p_\alpha)_{\alpha<\omega_1}$ is an uncountable antichain; then $\{dom(p_\alpha) | \alpha < \omega_1\}$ is an uncountable subset of $[T \times \omega]^{<\omega}$, so we can apply the delta-systems lemma and assume that for each α, $dom(p_\alpha) = r \cup d_\alpha$, where $r, (d_\alpha)_{\alpha<\omega_1}$ are pairwise disjoint. Let us also assume that the d_α all have the same cardinality k. There are only countably many possibilities for the values of the p_α on r, so we assume that all the conditions agree on r. So for any $\alpha, \alpha' < \omega_1$, there is $t \in d_\alpha$, $t' \in d_{\alpha'}$ and $n \in \omega$ such that $p_\alpha \cup p_{\alpha'}$ is not order preserving on $\{(t,n),(t',n)\}$, whence in particular t and t' are comparable in the tree order. As any node of the tree has only countably many predecessors in the tree order, by thinning out $(p_\alpha)_{\alpha<\omega_1}$ we can further assume that for all $\alpha, \alpha' < \omega_1$, there are $t \in d_\alpha$, $t' \in d_{\alpha'}$ such that $t <_T t'$ (find a function from ω_1 to ω_1 and consider closure points). Let us now enumerate the d_α as $t_\alpha^0, \ldots, t_\alpha^{k-1}$.

We know that all the conditions in the antichain have comparable nodes in their domain, we will now find a sufficiently coherent subset of conditions to get a branch through T. Enlarge (using Zorn's lemma or AC) the filter of co-initial subsets of ω_1 to an ultrafilter U (U contains only sets of size ω_1, i.e. U is uniform). For any $\alpha < \omega_1$, we have $\{ \beta < \omega_1 \mid \exists i, j \;\; t^i_\alpha <_T t^j_\beta \} \in U$. So by finite additivity of U, for each α, there are i, j such that $\{ \beta < \omega_1 \mid \; t^i_\alpha <_T t^j_\beta \} \in U$. Moreover, there is an uncountable set I and i, j such that the above holds for all $\alpha \in I$. So for any $\alpha, \alpha' \in I$, as elements of U have non-empty (in fact large) intersection, there is β such that $t^i_\alpha <_T t^j_\beta$ and $t^i_{\alpha'} <_T t^j_\beta$, so t^i_α and $t^i_{\alpha'}$ are comparable and $(t^i_\alpha)_{\alpha \in I}$ is an uncountable branch through T.

□

Now we can prove fact 5.6. We build P as the finite support iteration of $(P_k)_{k\in\omega}$. Let $s^0_\alpha = s_\alpha$; P_0 is the forcing coding this sequence of reals into a specializing function for T. At stage n, we have added a specializing function F_n; let s^{n+1}_α be a real coding F_n restricted to $T \upharpoonright \omega \cdot (\alpha + 1) \times \omega$. P_{n+1} is the forcing for coding the sequence $(s^n_\alpha)_{\alpha<\omega_1}$. Let r be a real coding all reals $(s^k_0)_{k\in\omega}$; we check by induction on $\eta \leq \omega_1$ that r has the property promised in 5.6: assume that for all k, $s^k_\eta \in M$; then for all k, F_k restricted to $T \upharpoonright \omega{\cdot}(\eta{+}1){\times}\omega$ is contained in M. Let $t \in T_{\omega\cdot(\eta+1)} \cap M \neq \emptyset$; by the continuity of the F_k and by lemma 5.7, $n \in s^k_{\eta+1}$ exactly if $sup(\{F(t') | t' <_T t\}) \in Q_0$; so $s^k_{\eta+1} \in M$ for all k. For limit η, using $(s^k_\xi)_{k\in\omega,\xi<\eta}$ we can decode F_k on $T \upharpoonright \omega \cdot \eta$ inside M, and therefore as before $s^k_\eta \in M$ for each k.

□

5.3 An equiconsistency

5.9 Theorem. *"$\mathbf{\Sigma^1_3}$(ccc)-absoluteness together with ω_1 inaccessible to reals" has the consistency strength of a lightface Σ^1_2-indescribable.*

Proof. We work in L: let κ denote $\omega_1{}^V$ and observe κ is inaccessible. Assume that the reflection property fails: there is $X^\star \in L_{\delta^\star}$ such that for some first order formula Φ (with parameter in L_κ, which we supress), $\langle L_\kappa, X^\star\rangle \models \forall A \Phi(X^\star, A)$, but for all $\xi < \kappa$ there is A such that $\langle L_\xi, X^\star \upharpoonright \xi, A\rangle \models \neg\Phi(X^\star \upharpoonright , A)$. We now define a tree T and its ordering $\leq_T$:

Elements of T are tuples (β, X), where $\beta < \kappa$ and $X \in {}^\delta 2$, for some δ, and

1. $L_\beta = h^{L_\beta}_{\Sigma_1}(|X| \cup X)$ (in particular, $X \in L_\beta$)

2. $X \upharpoonright |X| = X^\star \upharpoonright |X|$

3. for all $\xi \leq domX$, there is $A \in L_\beta$ such that $\langle L_\xi, X \upharpoonright \xi, A\rangle \models \neg\Phi(X \upharpoonright \xi, A)$.

Define $(\beta, X) \leq_T (\bar{\beta}, \bar{X}) \iff X \leq_L \bar{X}$ and there is a Σ_1-elementary embedding $\sigma : L_\beta \to L_{\bar{\beta}}$ such that $\sigma(X) = \sigma(\bar{X})$ and $crit(\sigma) \geq |X|$. This can be motivated by observing that branches correspond to a failure of reflection, as will become clear in a moment.

Let's check $\leq_T$ is a tree order. Clearly, $\leq_T$ is transitive and reflexive. Also, $\leq_T$ is antisymmetric: let (β, X) and (β', X') be a counterexample. As $X = X'$, the embedding witnessing $(\beta, X) \leq_T (\beta', X')$ shows L_β is isomorphic to $h^{L_{\beta'}}_{\Sigma_1}(|X'| \cup \{X'\})$, but the latter is just $L_{\beta'}$, by item 1 above. It remains to check that any two predecessors of a node are comparable: say (β, X), $(\beta', X') \leq_T (\bar{\beta}, \bar{X})$, as witnessed by embeddings σ and σ'. Without loss of generality assume $X \leq_L X'$, whence also $|X| \leq |X'|$. So $ran(\sigma) = h^{L_{\bar{\beta}}}_{\Sigma_1}(|X| \cup \{\bar{X}\}) \subseteq h^{L_{\bar{\beta}}}_{\Sigma_1}(|X'| \cup \{\bar{X}\}) = ran(\sigma')$, whence $(\sigma')^{-1} \circ \sigma$ is a well-defined elementary embedding and so $(\beta, X) \leq_T (\beta', X')$.

We now show T is a κ-Aronszajn tree, that is its level have size less than κ, it has height κ but no cofinal branch (i.e. linearly ordered subset of type κ). First observe that for a node $(\bar{\beta}, \bar{X})$ of T and a cardinal $\alpha \leq \beta$, there is exactly one predecessor node of cardinality α. Existence: look at the transitive collapse L_β of $h^{L_{\bar{\beta}}}_{\Sigma_1}(\alpha \cup \{\bar{X}\})$ and let X denote the image of $\bar{X}$ under the collapsing map (let σ denote the inverse of this map). Then $|X| = \alpha$, so $L_\beta = h^{L_\beta}_{\Sigma_1}(|X| \cup \{X\})$. Item 3 holds for $(\bar{\beta}, \bar{X})$, so by a Skolem hull argument, it also holds for (β, X). So $(\beta, X) \in T$. If $\alpha < |\beta|$, $X \leq_L \bar{X}$, and σ witnesses $(\beta, X) \leq_T (\bar{\beta}, \bar{X})$. If $\alpha = |\beta| = |X|$, by item 1, $X = \bar{X}$ and $\beta = \bar{\beta}$. Uniqueness: say (β, X), $(\beta', X') \leq_T (\bar{\beta}, \bar{X})$, and $\alpha = |\beta| = |\beta'|$. Then both $\langle L_\beta, X\rangle$ and $\langle L_{\beta'}, X'\rangle$ are ismorphic to $h^{L_{\bar{\beta}}}_{\Sigma_1}(\alpha \cup \{\bar{X}\})$, so they are identical. As a corollary we obtain that if $(\beta, X) \in T$ and $|\beta| = \omega_\alpha$, then the height of (β, X) in T is exactly α. So $T \upharpoonright \alpha \subseteq L_{\omega_\alpha}$ and T is a κ tree. For any $\alpha < \kappa$, let $X := X^\star \upharpoonright \omega_\alpha$ and let L_β be the transitive collapse of $H := h^{L_\kappa}_{\Sigma_1}(\omega_\alpha \cup \{X\})$. It is easy to check that $(\beta, X) \in T$ (for item 3, observe that $domX = \omega_\alpha \in H$) and we have seen its height is exactly α, so T has height κ.

5.10 Lemma. *T does not have a cofinal branch in V.*

Proof. Else, let $(\beta(\alpha), X(\alpha))_{\alpha<\kappa}$ be such a branch, let $\sigma^{\bar{\alpha}}_\alpha : L_{\beta(\alpha)} \to_{\Sigma_1} L_{\beta(\bar{\alpha})}$ be the embedding witnessing $(\beta(\alpha), X(\alpha)) \leq_T (\beta(\bar{\alpha}), X(\bar{\alpha}))$. A simple argument involving the Σ_1-definable Skolem function shows that for $\alpha < \alpha' < \bar{\alpha}$, $\sigma^{\bar{\alpha}}_{\alpha'} \circ \sigma^{\alpha'}_\alpha = \sigma^{\bar{\alpha}}_\alpha$. As κ has uncountable cofinality, the direct limit of this chain of models is well-founded and a model of $V = L$, therefore isomorphic to some L_δ. Each $L_{\beta(\alpha)}$ is Σ_1-elementarily embeddable into L_δ via a map that is the identity on $|\beta(\alpha)|^L$, and all the $X(\alpha)$ are mapped to one X_0 which must therefore end-extend $X^\star$ (in the sense that $X_0 \restriction \kappa = X^\star$). So $\delta > \kappa$ (as $X_0 \in L_\delta$). By elementarity (and condition 3 in the definition of T), there is $A \in L_\delta$ such that $\langle L_\kappa, X_0 \cap \kappa, A\rangle \models \neg\Phi(X_\delta \cap \kappa, A)$, contradiction. $\Box$

Let's go back to working in L again, for yet a little while. T is not pruned (there are dying branches and branches that don't split), and T needn't even have unique limit nodes (in the sense that for t and t' at a limit level T_λ, if t and t' have the same predecessors, then $t = t'$). The latter shortcoming has to be remedied, and this is accomplished easily by replacing T by T', where $T'_{\alpha+1} = T_\alpha$ for any $\alpha < \kappa$, while for limit ordinals λ we set $T'_\lambda = \{pred(t) | t \in T_\lambda\}$. T' carries the obvious order ($t \leq_T z'$ exactly if either $t \subseteq t'$ or $t \in t'$ or $t \subseteq pred(t')$ or $t \leq_T t'$).

Pick E such that

1. $\langle \kappa, E\rangle \cong \langle L_{\delta^\star}, \in\rangle$ and
2. $X^\star(\xi) = 1 \iff (\xi + 1)\ E\ \emptyset$.

Define $C := \{\xi < \kappa | \xi \text{ is a cardinal and } \langle L_\xi, X^\star \cap \xi, E \cap \xi\rangle \prec \langle L_\kappa, X^\star \cap \xi, E \cap \xi\rangle\}$. By inaccessibility of κ this is a *cub* set.

Let C be enumerated as $(c_\xi)_{\xi<\kappa}$. Now we work in V: let s_ξ code (via the transitive collapse) $(T_{\xi+1}, X^\star \restriction c_\xi, E \cap c_\xi)$. Now we apply the forcing just described (fact 5.6) to code the sequence $S = (s_\xi)_{\xi<\omega_1}$ into a single real r, using T.

Now let $\beta < \kappa$ and $L_\beta[r]$ be a model of $ZF^- \wedge \exists\omega_1$, and say $\alpha = \omega_1^{L_\beta[r]}$. We claim that for some $\xi \leq \alpha$, there is $x \in L_\beta$ such that for all $a \in L_\beta$, $\langle L_\xi, x, a\rangle \models \Phi(x, a)$, i.e. that from the point of view of L_β, reflection occurs before ω_1. Assume otherwise; we show how to recursively reconstruct $(s_\xi)_{xi\leq\alpha}$ inside $L_\beta[r]$. This is a contradiction as $s_\alpha \in L_\beta[r]$ means α is countable in that model. We construct $(s_\xi)_{xi\leq\eta}$ by recursion on $\eta \leq \alpha$. $s_0 \in L_\beta[r]$ is immediate. Now say $\eta = \gamma + 1$: so $(s_\xi)_{xi\leq\gamma} \in L_\beta[r]$, so $T \restriction \gamma + 2 \in L_\beta[r]$, so using the specializing functions on that tree we get $s_\gamma + 1 \in L_\beta[r]$. In the remaining case, where η is limit: by induction hypothesis, $(s_\xi)_{\xi<\eta} \in L_\beta[r]$. So $T \restriction \eta = \bigcup_{\xi<\eta} T_{\xi+1}$ is in $L_beta[r]$, and in fact, all of its elements are countable there. Likewise, $X^\star \restriction c_\eta, E \cap c_\eta \in L_\beta[r]$. $Ecapc_\eta$ is of course a well founded relation, and by the definition of C and elementarity, it is isomorphic to some $L_{\delta^\star}$ such that $X^\star \restriction c_\eta \in L_{\delta^\star}$ and $\delta^\star < \beta$. So since $c_\eta \geq \omega_\eta$, $X \restriction \omega_\eta \in L_\beta$ (to be sure we didn't use any information not accessible in $L_\beta[r]$, we feel we should mention the triviality that $(\omega_\eta)^L = (\omega_\eta)^{L_\beta}$). By assumption, there is $a \in L_\beta$ such that $L_{\omega_\eta} \models \neg\Phi(X^\star \restriction \omega_e ta, a)$. Therefore, by definability of

Σ_1-Skolem function and replacement in L_β, we can look at the collapse $L_{\beta^\star}$ of $h^{L\beta}_{\Sigma_1}(\omega_e ta \cup \{X^\star \restriction \omega_e ta\})$, $\beta^\star < \beta$. Clearly, $(\beta^\star, X^\star \restriction \omega_e ta) \in T$ (and its height is at least η; the only thing of substance to check for membership in T is item 3 in the definition of T). As all predecessor nodes of T are countable in $L_\beta[r]$, $\leq_T$ on $T \restriction \eta+1$ (which involves finding an embedding of structures) is absolute for $L_\beta[r]$ (see lemma 1.6). So $L_\beta[r]$ can use $(\beta^\star, X^\star \restriction \omega_e ta) \in T$ to find a branch though $T \restriction \eta$ and by continuity of the ranking functions, $s_\eta \in L_\beta[r]$.

So we have found, after forcing with a *ccc* partial order, a real r with the Σ^1_3 property

$$\forall\beta < \omega_1, \text{ if } L_\beta[r] \vDash ZF^- \wedge \exists\omega_1, \text{ then}$$
$$\exists\alpha \leq (\omega_1)^{L_\beta[r]} \text{ such that } (L_\alpha \vDash \exists X \forall A \Phi(X, A))^{L\beta}$$

So we may assume that r is in the ground model. But since ω_1 is inaccessible to reals, we may look at $\beta := (\omega_2)^L[r]$. By the above, for some $\alpha \leq (\omega_1)^{L_\beta[r]}$, $(L_\alpha \vDash \exists X \forall A \Phi(X, A))^{L\beta}$, and therefore $(L_\alpha \vDash \exists X \forall A \Phi(X, A))^L$. This completes one direction of the proof.

For the other direction, assume κ is inaccessible and lightface Π^1_2 - indescribable. We show that after gently collapsing κ to ω_1, $\mathbf{\Sigma^1_3}(ccc)$-absoluteness holds and κ is inaccessible to reals. The latter is clear, as any real in the extension can be absorbed into an intermediate model where κ is still inaccessible.

In $V^{Coll(\omega,\kappa)}$, let P be a *ccc* (i.e. $\kappa - cc$) p.o. that forces a $\Sigma^1_3(r)$ statement $\phi(r)$, $r \in V^{Coll(\omega,\{\alpha\})}$, for some $\alpha < \kappa$. Firstly, we can assume that $P \cong \kappa$: write $\phi(r)$ as "there is a real x such that a certain tree $T(x)$ on ω_1 (which is $\Delta_1(x, r, \omega_1)$) is well-founded". So $\Vdash_P$ "$\exists\dot{x}$ s.t. $T(\dot{x})$ is well-founded". As P has the $\kappa - cc$, there is $\xi < \kappa^+$ such that $\Vdash_P rank(T(\dot{x})) < \check{\xi}$. Thus there is a name $\dot{F}$ for a ranking function on $T(\dot{x})$, $\dot{F} \cong \kappa$. Now $\Vdash$"$\dot{F}$ is a ranking function on $T(\dot{x})$" is a first order statement over the structure $\langle P, \dot{F}, \dot{x}\rangle$ and thus also holds for an elementary submodel P' of size κ, containing $\dot{F}$ and $\dot{x}$. This proves we can assume $P \cong \kappa$. For the moment, we work in $W := V^{Coll(\omega,\{\alpha\})}$. We have that for $Q := Coll(\omega, \kappa) * P$, $Q \Vdash \phi(r)$, Q is of size κ and has the κ-cc. The fact that there exists such a Q can be expressed in a Σ^1_2 way, over V_κ: as $Q \cong \kappa$, w.l.o.g. we may assume $Q \subset V_\kappa$. Let $\Psi(A, Q)$ be first order in the predicates Q, A, saying "if A is an antichain of Q, A is equal to some set", and let $\Theta(Q, r)$ be first order in the predicate Q and parameter r saying "$\Vdash \phi(r)$". We have that $V_\kappa \vDash \exists Q \quad \forall A \quad \Psi(A, Q) \wedge \Theta(Q, r)$. By fact 5.2, κ still is Σ^1_2-indescribable (in W) so there is an inaccessible $\xi < \kappa$ reflecting this second order statement. So there is $Q \cong \xi$ s.t. $V_\xi \vDash$"$\Vdash_Q \phi(r)$", but as $V_\xi \vDash \forall A \Theta(Q, A))$, Q has the ξ-cc and thus $\Vdash_Q \phi(r)$ is absolute between V_ξ and V: ϕ mentions only reals, and all reals of the extension via Q have names in V_ξ; and once all quantification over names is removed from $\Vdash_Q \phi(r)$, only quantifiers ranging over $Q \subset V_\xi$ remain. Thus, in W, there is $Q \cong \xi$ forcing $\phi(r)$. So by corollary 1.30, $\phi(r)$ is forced by the gentle collapse of κ over W, and this means that, if $\dot{r}$ is some $Coll(\omega, \kappa)$-name for r, $Coll(\omega, \kappa) \sim Coll(\omega, \{\alpha\}) * Coll(\omega, \kappa)$ forces $\phi(\dot{r})$ over the ground model. So $\phi(r)$ holds in $V^{Coll(\omega,\kappa)}$, whence $\mathbf{\Sigma^1_3}$-absoluteness holds between this model and any subsequent *ccc* extension.

$\square$

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