Regularity properties for sets of reals

David Schrittesser

Kurt Gödel Research Center
University of Vienna

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What we talk about when we talk about $\mathbb{R}$

Joke

What's a set theorist? Someone who doesn't know what the real numbers are.

What we talk about when we talk about $\mathbb{R}$

Most of what follows works for any uncountable Polish space $\mathfrak{X}$ (they are all Borel isomorphic).

When I say "the reals" I might mean any of these spaces:

- $\omega^\omega = \mathbb{N}^\mathbb{N}$, the Baire space. $\omega^{<\omega}$ = the finite sequences.
- $2^\omega = 2^\mathbb{N}$, the Cantor space. $2^{<\omega}$ as above.
- $\mathbb{R}$
- $[\omega]^\omega$, the infinite subsets of $\omega = \mathbb{N}$. $[\omega]^{<\omega}$ = finite subsets. Identify $[\omega]^\omega$ with
  - $\omega^{\uparrow\omega}$, the increasing functions $f \colon \omega \to \omega$ (isomorphic to ω^ω);
  - or with a (G_δ) subset of $\mathcal{P}(\mathbb{N}) = 2^\omega$ (characteristic functions).

Some constructions *do* require to work in a specific space.

Projective Hierarchy

Let $\mathfrak{X}$ be a Polish space.

$$\mathbf{\Sigma}^1_1 = \text{analytic} = \text{continuous images of Borel sets}$$

$$\mathbf{\Pi}^1_n = \text{all complements of } \mathbf{\Sigma}^1_n \text{ sets;}$$

$$\mathbf{\Sigma}^1_{n+1} = \text{all continuous images of } \mathbf{\Pi}^1_n \text{ sets;}$$

$$\mathbf{\Delta}^1_n = \mathbf{\Pi}^1_n \cap \mathbf{\Sigma}^1_n$$

$$\mathbf{\Sigma}^1_\omega = \bigcup_{n\in\omega} \mathbf{\Sigma}^1_n$$

Projective Hierarchy, logically

In ω^ω, we may also see the projective hierarchy in terms of definable sets, by counting quantifiers over "reals":

Call $\Psi(x)$ a $\mathbf{\Sigma}^1_n$ formula iff it is of the form

$$\Psi(x) \equiv (\exists f_1 \in \omega^\omega)(\forall f_2 \in \omega^\omega)\dots(\overset{\exists}{\forall} f_n \in \omega^\omega)\Phi(f_1, f_2, \dots, f_n, x, g)$$

where Φ is in the language of arithmetic (say), contains only quantifiers over natural numbers and has $g \in \omega^\omega$ as a parameter.

A set A is $\mathbf{\Sigma}^1_n$ iff it is definable by a $\mathbf{\Sigma}^1_n$ formula $\Psi(x)$, i.e.

$$A = \{x \in \omega^\omega \mid \Psi(x)\}$$

Similarly for $\mathbf{\Pi}^1_n$ etc.

Something similar can be done for any Polish $\mathfrak{X}$.

Our main examples: measure and category

Two equivalent forms of measurability and the Baire-property:

$X \subseteq \mathfrak{X}$ is measurable (LM)

- $\iff$ $X = B \Delta N$ where B Borel (or F_σ, or G_δ), N null
- $\iff$ for every non-null closed C there is a non-null closed $C' \subseteq C$ s.t. $C' \cap X$ is null or $C' \subseteq^* X$

$X \subseteq \mathfrak{X}$ has the Baire property (BP) :

- $\iff$ $X = B \Delta M$, where B is Borel (or open), M meager
- $\iff$ $\forall p$ basic open $\exists q \subseteq p$ basic open s.t. $q \cap X$ is meager or $q \subseteq^* X$

Example: Marczewski-measurable

Another regularity property is being Marczewski-measurable.

Definition

$X \subseteq \mathfrak{X}$ is *Marczewski-measurable* $\iff$
for every perfect C there is a perfect $C' \subseteq C$ such that $C' \subseteq X$ or $C' \cap X = \emptyset$.

- Measurable sets form a σ-algebra containing the analytic sets.
- There's an associated σ-ideal of *Marczewski-null* sets:

Definition

$X \subseteq \mathfrak{X}$ is *Marczewski-null* $\iff$
for every perfect C there is a perfect $C' \subseteq C$ such that $C' \cap X = \emptyset$.

(same as being countable for Borel sets)

Example: completely Ramsey

Another regularity property is being completely Ramsey.

Work in $\mathfrak{X} = [\omega]^\omega$ (the infinite subsets of ω).

For $s \in [\omega]^{<\omega}$ and $U \in [\omega]^\omega$, let

$$[s, U] = \{X \in [\omega]^\omega \mid s \subseteq_{end} X \subseteq s \cup U\}.$$

$s \subseteq_{end} X$ means $s \subseteq X$ and $\max(s) < \min(X \setminus s)$.

- These are perfect (in the subspace topology)
- They form a basis for the *Ellentuck topology* on $\mathfrak{X}$

Example: completely Ramsey

Definition

$X \subseteq \mathfrak{X}$ is *completely Ramsey* $\iff$ X has the Baire-property with respect to the Ellentuck topology.

Two equivalent forms:

- for any $s \in [\omega]^{<\omega}$ and $U \in [\omega]^{\omega}$, there is $V \in [U]^{\omega}$ s.t. either $[s, V] \subseteq X$ or $[s, V] \cap X = \emptyset$.
- $\forall [s, U]\ \exists [t, V] \subseteq [s, U]$ s.t. either $[t, V] \subseteq X$ or $[t, V] \cap X = \emptyset$.

- The completely Ramsey sets form a σ-algebra containing the analytic sets (Silver 1970)...
- ...with an associated σ-ideal of Ramsey-null sets

Regularity notions: a general approach

Say $P \subseteq \mathcal{P}(\mathfrak{X})$ consists of closed subsets of $\mathfrak{X}$. Think of P as a partial order, ordered by $\subseteq$.

We can define P-regular (also called P-measurable):

Definition

For $X \subseteq \mathfrak{X}$

1. X is in $I_P \iff \forall C \in P \quad \exists C' \in P$ s.t. $C' \subseteq C$ and $C' \cap X = \emptyset$.
2. The ideal of P-null sets (N_P) is the σ-ideal generated by I_P.
3. X is P-regular $\iff \forall C \in P \quad \exists C' \in P$ s.t. $C' \subseteq C$ and ($C' \subseteq^* X$ or $C' \subseteq^* \mathfrak{X} \setminus X$).

($\subseteq^*$ denotes inclusion modulo a P-null set)

The examples revisited (say $\mathfrak{X} = \omega^\omega$)

1. $P = \mathbf{C}$, the basic clopen sets:
   - X is in $I_P \iff X$ is nowhere dense $\iff$ $\forall C \in \mathbf{C} \quad \exists C' \in \mathbf{C}$ s.t. $C' \subseteq C$ and $C' \cap X = \emptyset$.
   - $X \in N_P \iff X$ is meager.
   - X is $\mathbf{C}$-regular $\iff X$ has BP $\iff$ $\forall C \in \mathbf{C} \quad \exists C' \in \mathbf{C}$ s.t. $C' \subseteq C$ and $(C' \subseteq^* X$ or $C' \subseteq^* \mathfrak{X} \setminus X)$.
2. X is LM $\iff \forall C \in \mathbf{B} \quad \exists C' \in \mathbf{B}$ s.t. $C' \subseteq C$ and $(C' \subseteq^* X$ or $C' \subseteq^* \mathfrak{X} \setminus X)$
   ($P = \mathbf{B}$, the positive closed sets)
3. X is compl. Ramsey $\iff \forall C \in \mathbf{R} \quad \exists C' \in \mathbf{R}$ s.t. $C' \subseteq C$ and $(C' \subseteq^* X$ or $C' \subseteq^* \mathfrak{X} \setminus X)$
   ($P = \mathbf{R}$, the Ellentuck basis sets)

In the last two cases $\subseteq^*$ can be replaced by $\subseteq$ (by a special property of these P, called fusion).

In practice we usually consider $\mathfrak{X} = \omega^\omega$ or $\mathfrak{X} = 2^\omega$.

Given a tree $T \subseteq \omega^{<\omega}$ (resp. 2^ω), let

$$[T] = \{x \in \omega^\omega \mid \text{all initial segments of } x \text{ belong to } T\}.$$

Then $T \mapsto [T]$ is a bijection between (pruned) trees and closed sets.

Thus we can assume P in the above consists of perfect trees, ordered by $\subseteq$.

Any partial order $\langle P, \leq_P \rangle$ is also called a *forcing*.
Forcings like the the ones discussed above are called “arboreal”.

More examples

Arboreal forcing	Regularity property	P consists of...
S (Sacks)	Marczewski	perfect trees on $2^{<\omega}$
V (Silver)	Doughnut	uniform perfect trees on $2^{<\omega}$
M (Miller)		superperfect tree on $\omega^{<\omega}$ (infinite splitting)
L (Laver)		every node above stem splits infinitely
B (Random)	LM	perfect T s.t. $\mu([T]) > 0$
C (Cohen)	BP	full tree above stem (basic clopen)
R (Mathias)	compl. Ramsey	Ellentuck basis sets

Other examples: **D** (Hechler), Matet, eventually different forcing, . . .

Quick Overview:

For the first part of this talk, we shall not distinguish between notions of regularity.

- all analytics $\mathbf{\Sigma}^1_1$ sets are regular (LM, BP, Ramsey, ...).
- This is all that can be proven in ZFC:
  - There is a model of ZFC, L, where regularity fails for $\mathbf{\Delta}^1_2$ sets (1930ies)
  - Assuming an inaccessible cardinal, we can find model of ZFC where all projective sets are regular (Solovay's model, 1970).
  - The inaccessible can be shown to be necessary for this (Shelah 1984).
  - regularity can be characterized by transcendence over inner models (such as L); (Solovay 1979... ongoing research).
- Stronger axioms (projective determinacy, existence of Woodin cardinals) *prove* that all projective sets are regular.

Gödels constructible universe

L is an *inner model* of set theory, that is a definable class.

- minimal: $L \subseteq M$ for any class model M of ZFC
- absolute: If $M \subseteq N$ are class models of ZFC, $L^M = L^N$
- GCH holds in L, i.e. $2^\kappa = \kappa^+$ for all cardinals κ
- L has a definable global well-order $\leq_L$
- $\leq_L$ is a Σ^1_2-good well-order on 2^ω

Given a set X, we can form $L(X)$

- $L(X)$ is the least model M of ZF s.t. $X \subseteq M$
- $L(X)$ is also absolute
- $L(X)$ needn't be a model of AC (can fail in $L(\mathcal{P}(\omega)) = L(\mathbb{R})$)
- If $x \in 2^\omega$, $L[x] = L(\{x\})$ has the same nice properties as L.

Recall that a $X \subseteq \mathfrak{X}$ is a Bernstein set $\iff$ both X and $\mathfrak{X} \setminus X$ intersects every uncountable closed set.

Fact

For any standard notion of regularity, a Bernstein set is not regular.

Proof.

Being P-measurable requires to be disjoint from or contain some $C \in P$; and C is perfect. $\square$

Theorem (Gödel)

In L, $\leq_L$ yields a $\mathbf{\Sigma}^1_2$-good well-order of ω^ω, i.e. the set of $(x, \vec{z})$ such that $\vec{z}$ "enumerates" $\{y \in \omega^\omega \mid y \leq_L x\}$ is $\mathbf{\Sigma}^1_2$.

Corollary

In L, there is a $\mathbf{\Delta}^1_2$ Bernstein set, so the $\mathbf{\Delta}^1_2$ sets aren't regular.

Intro: Large cardinals - 'Strong axioms of infinity"

- General form: there is a cardinal with such and such properties.
- Existence of large cardinals is NOT provable in ZFC
- They form a hierarchy of stronger and stronger axioms...
- ...proving more about V and allowing to construct more models (inner models or forcing extensions).

Examples:

$$\text{inaccessible} < \text{Mahlo} < \text{measurable} < \text{Woodin cardinals}$$

where $<$ means, e.g.

$$\text{Con}(\text{ZFC}+\text{“}\exists\text{ a Mahlo”}) \Rightarrow \text{Con}(\text{ZFC}+\text{“}\exists\text{ an inaccessible”})$$

Note: $\text{Con}(\text{ZFC}+\text{“}\exists\text{ an inaccessible”}) \Rightarrow \text{Con}(\text{ZFC})$

Intro: Forcing

Let $\langle P, \leq_p \rangle$ be a quasi-order (transitive, reflexive), and M be a model of ZFC.

Definition

$D \subseteq P$ is dense $\iff \forall p \in P \, \exists q \in D \; q \leq_P p$.
$G \subseteq P$ is (M, P)-generic $\iff \forall D \in M$ s.t. $D \subseteq P$ dense, $D \cap G \neq \emptyset$.

- We can form an extension, called M^P or $M[G] \supset M$ containing an (M, P)-generic G
- M can be e.g. L or $V =$ the universe of all sets
- Of course, this involves "meta-mathematical" fine-points
- By careful choice of P, we may be able to "force" a statement Φ to hold in $M[G]$
- We have found a model of set theory where Φ holds (and so ZFC does not prove $\neg\Phi$).

Examples: Forcing

If we force over M with one of the P mentioned earlier, we obtain a *P-generic real r* in $M[G]$ s.t.

$$\{r\} = \bigcap\{C^{M[G]} \mid C \in G\}$$

(remember $C = [T]$ is a closed set, so it has a “description” T which can be interpreted in $M[G]$) and $G = G_r = \{C \in P \mid r_G \in C\}$.

Vice versa, a real r such that G_r is P-generic over M is called a P-generic over M (or a Cohen real, random real, Sacks real etc).

Fact

If r is P-generic over $M \Rightarrow$ for any Borel $B \in M$ which is co-P-null in M, $r \in B^{V[G]}$.

$\Leftarrow$ holds for random and Cohen, but fails e.g. fails for Silver and Sacks forcing.

Transcence principles for $n = 2$

Theorem (Judah-Shelah 1989)

$\mathbf{\Delta}^1_2(\mathbf{C}) \iff (\forall r \in \omega^\omega)(\exists s \in \omega^\omega)$ s.t. s is $(L[r], \mathbf{C})$-generic.
$\mathbf{\Delta}^1_2(\mathbf{B}) \iff (\forall r \in \omega^\omega)(\exists s \in \omega^\omega)$ s.t. s is $(L[r], \mathbf{B})$-generic.

More theorems (Solovay, 1970; Brendle, 1999)

$\mathbf{\Sigma}^1_2(\mathbf{C}) \iff (\forall r \in \omega^\omega)\{$ s.t. s is not $(L[r], \mathbf{C})$-generic $\}$ co-meager.
$\mathbf{\Sigma}^1_2(\mathbf{B}) \iff (\forall r \in \omega^\omega)\{$ s.t. s is $(L[r], \mathbf{B})$-generic $\}$ co-null.
$\mathbf{\Sigma}^1_2(\mathbf{S}) \iff (\forall r \in \omega^\omega)(\exists s \in \omega^\omega)$ s.t. $s \notin L[r]$.
$\mathbf{\Sigma}^1_2(\mathbf{M}) \iff (\forall r \in \omega^\omega)(\exists s \in \omega^\omega)$ *unbounded* over $L[r]$.
$\mathbf{\Sigma}^1_2(\mathbf{L}) \iff (\forall r \in \omega^\omega)(\exists s \in \omega^\omega)$ *dominating* over $L[r]$.

- Ikegame (2012) showed there is a uniform-in-P version of these, involving an ideal I^*_P and a notion of *quasi-generic* reals (e.g. dominating; same as *generic* for Cohen and random).
- He also showed there are analogues (replacing L with *core models*) for $n > 2$ *assuming relatively large cardinals.*

Solovay's model

Theorem (Solovay, 1970; Mathias, 1977; others)

Assume there is an inaccessible cardinal. Then there is a partial order Coll such that in $V[G]$, every projective set is regular.

Also: in $L(\mathbb{R})^{V[G]}$, all sets are regular (AC fails).

Rough sketch of proof 1

Let κ be inaccessible. Let Coll $=$ set of finite partial functions $f\colon \kappa \times \omega \to \kappa$ such that $f(\alpha, n) < \alpha$; ordered by $\subseteq$ (extension).

It's easy to see that κ becomes ω_1 in $V[G]$.

In $V[G]$, let $A \subseteq \mathfrak{X}$, defined by the Σ^1_n formula $\Psi(x)$, and $C \in P$ be given. We must find $C' \subseteq C$ such that $C' \subseteq^* A$ or $C' \cap A \in N_P$.

For simplicity assume $C = \mathfrak{X}$ (and that $\Psi(x)$ is without parameter).

Basic fact

When forcing, P may be replaced by its *Stone space*, St(P), a complete Boolean algebra. For every formula $\Phi(x)$, we may define $\|\Phi(x)\| \in \mathrm{St}(P)$ such that

$$\Phi(x) \text{ holds in } M[G] \iff \|\Phi(x)\| \in G$$

Rough sketch of proof 2

Property A of Coll

In $V[G]$, Coll has added a co-P-null set of $P \cap V$-generic reals $\subseteq C$ for every $C \in P$.

Pick a P-generic r and assume without loss of generality $r \in A$.
By a general fact, $\mathrm{St}(P)$ is a complete sub-algebra of St(Coll) (their Stone-spaces).

Property B of Coll

If A and B are two isomorphic small subalgebras of Coll, there is an automorphism f of St(Coll) such that $f[A] = B$.

By "mixing," it follows that $\|\Psi(r)\|^{\mathrm{Coll}} = \|\Psi(r)\|^{\mathrm{St}(P)} \in \mathrm{St}(P)$. As P is dense in $\mathrm{St}(P)$, we can find $C' \in P$ s.t. $C' \leq_P \|\Psi(r)\|$.
This means that every $(V \cap P, V)$-generic real $r' \in C'$ will satisfy $\Psi(r')$ in $V[G]$. By property A, we are done. $\square$ *(End of sketch.)*

Games and regularity

Recall the *Banach-Mazur* game $G_\omega(X)$, for $X \subseteq \omega^\omega$:
Players take turns picking basic open sets U_k in ω^ω, for $k \in \omega$, such that $U_{k+1} \subseteq U_k$. Player I wins if $\emptyset \neq \bigcap_k U_k \subseteq X$, Player II wins otherwise.

- I has a winning strategy $\Rightarrow$ X is co-eager on an open set.
- II has a winning strategy $\Rightarrow$ X is meager.

"Unfolded" version of the Banach-Mazur game shows:

$$\mathbf{\Pi}^1_n\text{-determinacy} \Rightarrow \mathbf{\Sigma}^1_{n+1}(\mathrm{BP})$$

Löwe (1996) describes in great generality how such games can be used to show:

Fact

$\mathbf{\Pi}^1_n$-determinacy $\Rightarrow$ $\mathbf{\Sigma}^1_{n+1}(P)$ for all our P.

This also gives a uniform proof that all $\mathbf{\Sigma}^1_1$ sets are regular.

The end of independence

Theorem (Martin-Steel, 1989)

Let $n \geq 1$. If there are n Woodin cardinals, $\mathbf{\Pi}^1_{n+1}$-determinacy holds, hence all $\mathbf{\Sigma}^1_{n+1}$ sets are regular.

Corollary (Martin-Steel, 1989)

If there are infinitely many Woodin cardinals, projective determinacy holds. Hence all projective sets are regular.

Thus, the question of separating regularity properties becomes vacuous if we agree to admit this large cardinal axiom as a de-iure axiom in set theory.

Unsatisfying consistency results

Theorem (?)

If there is a measurable cardinal, there is a model $L[U]$ in which there is a $\boldsymbol{\Delta}^1_3$-good well-order of ω^ω and $\boldsymbol{\Pi}^1_1$-determinacy holds.
So in $L[U]$ we have $\boldsymbol{\Sigma}^1_2(P) + \neg\boldsymbol{\Delta}^1_3(P)$ for all P.

But models with $\boldsymbol{\Sigma}^1_2(P) + \neg\boldsymbol{\Delta}^1_3(P)$ for all P can be found from just an inaccessible.

Theorem (Martin-Steel)

Let $n \geq 1$. If there are n Woodin cardinals, there is a model M_n in which there are n Woodin cardinals and a $\boldsymbol{\Delta}^1_{n+2}$-good well-order of ω^ω.
So in M_n we have $\boldsymbol{\Sigma}^1_{n+2}(P) + \neg\boldsymbol{\Delta}^1_{n+3}(P)$ for all P

Models with $\boldsymbol{\Sigma}^1_{n+2}(P) + \neg\boldsymbol{\Delta}^1_{n+3}(P)$ should also be obtainable from just an inaccessible (but currently aren't)!

Separating notions of regularity

- There are some ZFC implications, e.g. (Bartoszinski-Raisonnier)

$$\mathbf{\Sigma}^1_2(LM) \Rightarrow \mathbf{\Sigma}^1_2(BP)$$

- Initially the interest was separating LM from BP:
  - Shelah first separated BP and LM, by showing $\mathbf{\Sigma}^1_\omega(BP) \not\Rightarrow \mathbf{\Sigma}^1_\omega(LM)$.
  - For the other direction, Judah, Bagaria, and Woodin obtained partial results for $n < 4$.
  - In a joint work with Sy Friedman, I showed $\mathbf{\Sigma}^1_\omega(LM) \not\Rightarrow \mathbf{\Sigma}^1_\omega(BP)$.
- for $n < 4$, almost everything is solved for all above P (by the transcendence method) (Fischer-Friedman-Khomskii, 2013).

Separating category from measure: Amalgamation 1

Theorem 1 (Shelah 1984)

Any model of ZFC V has an extension $V[G]$ where every projective set has the Baire-property.

- Note that you don't need large cardinals!
- Idea: imitate Solovay's proof by building a forcing which has automorphisms (property B) for St($\mathbf{C}$), i.e. Cohen sub-algebras.
- To build such a forcing Shelah invented *amalgamation*

Theorem 2 (Shelah 1984)

If every Σ^1_3 set is measurable, ω_1 is inaccessible in L.

This is shown using a *rapid filter*.

Separating category from measure: Amalgamation 2

Theorem 1

Any model of ZFC V has an extension $V[G]$ where every projective set has the Baire-property.

Theorem 2

If every Σ^1_3 set is measurable, ω_1 is inaccessible in L.

Corollary

There is a model where every projective set has BP, but there is a projective set without LM.

Proof.

Let $V = L$ and there assume is no inaccessible. Go to the model $L[G]$ provided by the first theorem. By assumption $\omega_1^{L[G]}$ is not inaccessible in L, so in $L[G]$ there is a Σ^1_3 set which is not measurable. $\square$

Separating by consistency strength

Note: there are two types of notions of regularity:

1. those equiconsistent with ZFC (a model for $\Sigma^1_\omega(P)$ can be found from just ZFC, as for BP)
2. those equiconsistent with an inaccessible (a model for $\Sigma^1_\omega(P)$ *cannot* be found from just ZFC, as for LM)

Long standing open question

What is the situation for Mathias forcing $\mathbf{R}$ (i.e. for the notion of being completely Ramsey) and for Laver forcing $\mathbf{L}$?

Separating measure from category

Theorem (Friedman-S.)

Assume a Mahlo cardinal. Consistently, $\Sigma^1_\omega(\mathbf{B}) \wedge \neg\Delta^1_3(\mathbf{C})$

- $\Sigma^1_3(\mathbf{B}) \Rightarrow \omega_1$ is inaccessible in L (Shelah).
- This in turn implies $\Sigma^1_2(\mathbf{C})$, by one of the transcendence principles, see below.
- So the complexity of the irregular set is optimal.
- The proof uses a different amalgamation method and Jensen's "coding of the universe by a real" in a localized version.

ZFC Implications

The following implications hold in ZFC:

$$\begin{tikzcd}
\Gamma(\mathbf{B}) \arrow[r, Rightarrow] & \Gamma(\mathbf{V}) \arrow[rr, Rightarrow] & & \Gamma(\mathbf{S}) \\
\Gamma(\mathbf{R}) \arrow[r, Rightarrow] \arrow[ur, Rightarrow] & \Gamma(\mathbf{L}) \arrow[r, Rightarrow] & \Gamma(\mathbf{M}) \arrow[ur, Rightarrow] & \\
& & \Gamma(\mathbf{C}) \arrow[u, Rightarrow] \arrow[uul, Rightarrow] &
\end{tikzcd}$$

Let's call the above "Cichon's diagram for regularity".

$$\mathbf{\Sigma}^1_2(\mathbf{P}) = \mathbf{\Delta}^1_2(\mathbf{P}) \text{ for } P \in \{\mathbf{R}, \mathbf{L}, \mathbf{M}, \mathbf{S}\}$$

$$\Sigma^1_2(\mathbf{B}) \Rightarrow \Sigma^1_2(\mathbf{C}) \Rightarrow \Sigma^1_2(\mathbf{V}) \Rightarrow \Sigma^1_2(\mathbf{M})$$

Disclaimer: not a complete list.

$n < 4$ and the transcendence method

For $\mathbf{\Delta}^1_n$ and $\mathbf{\Sigma}^1_n$ when $n < 4$, Fischer-Friedman-Khomskii (2013) have found models for almost every situation not excluded by known ZFC implications.

They list a few open questions, among them:

Open question

$$\text{Does } \Delta^1_2(\mathbf{L}) \Rightarrow \Sigma^1_2(\mathbf{V})?$$

$n = \omega$: Generalizing Friedman-S.

Theorem (Brendle?)

There is a model (where AC fails) in which *all* sets are **V**-regular, but not all sets are **B**-regular.

The model is obtained by a standard forcing extension adding many Cohen reals to L (the "Cohen model"); (the status of the other notions is currently unclear in this model).

The general method to prove such results should be amalgamation, as in the following result:

Theorem (Laguzzi, 2012)

If there is an inaccessible, we can find a model where *all* sets are **V**-regular, but not all sets are P-regular for each of $P \in \{\mathbf{M}, \mathbf{B}\}$ (and ω_1 is inaccessible in L).

Observe that by Cichon's diagram, this determines which sets in the projective hierarchy are regular for all the listed notions.

$n = \omega$: Generalizing Friedman-S.

To use a amalgamation to separate P_0 from P_1-regularity, one needs to show they add *very different kind of reals*.

E.g. Random forcing adds no *unbounded reals*, while Cohen reals *are* unbounded over V.

Laguzzi found Silver forcing doesn't add *unreachable reals*, while Miller and Random reals *are* unreachable over V.

We can turn this into a projective result:

Theorem

Assume a Mahlo. There is a model $V[G]$ of ZFC in which we have

$$\Sigma^1_\omega(\mathbf{V}) + \neg\mathbf{\Delta}^1_3(\mathbf{M}) + \neg\mathbf{\Delta}^1_3(\mathbf{B})$$

Two more questions:

Let me conclude with two more open questions:

1. How can we "separate" other notions of regularity? E.g. for Silver and Sacks, if applicable, our method would yield a model of

$$\Sigma^1_\omega(\mathbf{S}) + \neg\mathbf{\Delta}^1_3(\mathbf{V})$$

2. Can we force, from modest large cardinal assumptions

$$\Sigma^1_7(\mathbf{C}) + \neg\Sigma^1_8(\mathbf{C})$$

   In a model with just the right finite number of Woodin cardinals, $\Sigma^1_7(\mathbf{P}) + \neg\mathbf{\Delta}^1_8(\mathbf{P})$ holds for all P. But something of the order of an inaccessible should suffice to find a model of it (a forcing extension).

Thank You!